Via Afrika Mathematics

Grade 11 Study Guide

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Via Afrika

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Study Guide

Via Afrika Mathematics

Grade 11

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Introduction to Via Afrika Mathematics Grade 11 Study Guide

Woohoo! You made it! If you're reading this it means that you made it through Grade 10, and are now in Grade 11. But I guess you are already well aware of that...

It also means that your teacher was brilliant enough to get the *Via Afika Mathematics Grade 11 Learner's Book*. This study guide contains summaries of each chapter, and should be used side-by-side with the Learner's Book. It also contains lots of extra questions to help you master the subject matter.

Mathematics – not for spectators

You won't learn anything if you don't involve yourself in the subject-matter actively. Do the maths, feel the maths, and then understand and use the maths.

Understanding the principles

- **Listen during class.** This study guide is brilliant but it is not enough. Listen to your teacher in class as you may learn a unique or easy way of doing something.
- **Study the notation, properly.** Incorrect use of notation will be penalised in tests and exams. Pay attention to notation in our worked examples.
- **Practise, Practise, Practise, and then Practise some more.** You have to practise as much as possible. The more you practise, the more prepared and confident you will feel for exams. This guide contains lots of extra practice opportunities.
- **Persevere.** We can't all be Einsteins, and even old Albert had difficulties learning some of the very advanced Mathematics necessary to formulate his theories. If you don't understand immediately, work at it and practise with as many problems from this study guide as possible. You will find that topics that seem baffling at first, suddenly make sense.
- **Have the proper attitude.** You can do it!

The AMA of Mathematics

ABILITY is what you're capable of doing.

MOTIVATION determines what you do.

ATTITUDE determines how well you do it.

“Give me a place to stand, and I will move the earth!” *Archimedes*

Chapter 1

Exponents and Surds

Overview

Chapter 1 Page 3 Exponents and surds	**Unit 1 Page 4**	• Exponents and Surds • Exponential equations
	Rational exponents	
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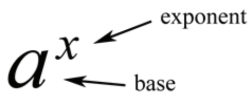
In this chapter we review the laws of exponents and exponential equations. When we've covered that, we will have a look at rational exponents and surds. You will also learn how to solve exponential equations, simplify surds and solve equations containing surds.

Unit 1

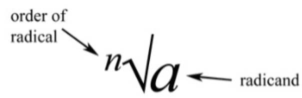
Rational exponents

1.1 Exponents and surds

- The exponent of a number tells you how many times the number has to be multiplied by itself.
- A surd is a number that cannot be simplified further to remove the root. They are irrational numbers.



- We always assume that a root without a number in front of it is a square root.
- The square root of a number a can be written as $\sqrt{a}$, or in exponential form $a^{1/2}$.
- The cube root of a number b can be written as $\sqrt[3]{b}$, or in exponential form $b^{1/3}$.
- The nth root of a number c can be written as $\sqrt[n]{c}$, or in exponential form $c^{1/n}$.
- In the expression $\sqrt[3]{64}$, the 3 is the order of the radical and 64 is the radicand. We read $\sqrt[3]{64}$ as the 3rd root of 64.



- In the exponential form, the base of the expression is the radicand.
- Remember, if we have the equation $\sqrt[n]{a} = a^{1/n}$ and we raise both sides to the power k, we have $(\sqrt[n]{a})^k = (a^{1/n})^k$, this simplifies to $\sqrt[n]{a}^k = a^{k/n}$.

Example 1

Simplify the following equations without using a calculator:

1 $27^{1/3} = (3^3)^{1/3}$

$= 3$

2 $(3\,\frac{3}{8})^{-2/3} = (\frac{27}{8})^{-2/3}$

$= (\frac{8}{27})^{2/3}$

$= [(\frac{2}{3})^3]^{2/3}$

$= (\frac{2}{3})^2 \quad = \frac{4}{9}$

3 $(4a^4b^{16}c^8)^{1/2} = (2^2a^4b^{16}c^8)^{1/2}$

$= 2a^2b^8c^4$

1.2 Exponential equations

- Some exponential equations have only one solution, while others have more.
- Remember, this can be generalised as:
  If $x^{a/b} = c$, where c is a constant, then
  - if a is odd, there is only one solution.
  - if a is even, there are two solutions. One will be positive, and one will be negative.

Example 2

Solve the following equations without using a calculator:

1 $x^{1/2} = 3$

$[x^{1/2}]^2 = 3^2$

$\therefore x = 3$

2 $x^{2/5} = 4$

$[x^{2/5}]^{5/2} = \pm [4]^{5/2}$

$x = \pm (2^2)^{5/2}$

$\therefore x = \pm 32$

Surds

2.1 Types of surds

- A surd is an irrational number and it contains a radical.
- We can use the following laws to help us simplify expressions:
  - o Product rule: $\sqrt[m]{a}.\sqrt[m]{b} = \sqrt[m]{a.b}$
  - o Quotient rule: $\sqrt[m]{a}/\sqrt[m]{b} = \sqrt[m]{a/b}$

  Note that these laws only apply to multiplication and division, and $a > 0$ and $b > 0$.
- When we simplify surds, we write the numbers as the product of perfect squares and other numbers, e.g. $\sqrt{8} = \sqrt{(4.2)} = 2\sqrt{2}$

Example 3

Simplify the following without using a calculator:

1 $2\sqrt{3} + 9\sqrt{3} - 16\sqrt{3} + 6\sqrt{3} = \sqrt{3}$

2 $2\sqrt{3} + 9\sqrt{2} - 16\sqrt{3} + 6\sqrt{2} = 15\sqrt{2} - 14\sqrt{3}$

3 $\sqrt{75} - \sqrt{18} = \sqrt{25.3} - \sqrt{9.2}$

$= \sqrt{25}\sqrt{3} - \sqrt{9}\sqrt{2}$

$= 5\sqrt{3} - 3\sqrt{2}$

4 $(\sqrt{48} + \sqrt{27})/\sqrt{75} = (\sqrt{16.3} + \sqrt{9.3})/\sqrt{25.3}$

$= (\sqrt{16}\sqrt{3} + \sqrt{9}\sqrt{3})/\sqrt{25}\sqrt{3}$

$= (4\sqrt{3} + 3\sqrt{3})/5\sqrt{3}$

$= 7\sqrt{3}/5\sqrt{3}$

$= \frac{7}{5}$

2.2 Multiplying and dividing surds

- To multiply or divide surds, we often need to use the distributive property:
  $a(b + c) = ab + bc$
- Some problems need to be solved by rationalisation. Rationalisation is the process where we convert a denominator/numerator with an irrational number to a rational number. We do this by multiplying the expression by the surd divided by itself.

Example 4

Simplify the following without using a calculator:

1 $5(2\sqrt{3} + 9) = 10\sqrt{3} + 45$

2 $(\sqrt{5} - 6)(2\sqrt{6} + 8) = 2\sqrt{30} - 12\sqrt{6} + 8\sqrt{5} - 48$

3 $(\sqrt{5} + 3)(\sqrt{5} - 3) = 5 - 9 = -4$

2.3 Equations with surds

- To solve equations that contain surds, we first have to remove the surd.
- To remove the surd, we have to raise both sides to the order of the radical, e.g. if we have to solve an equation with a square root, we first square both sides.
- Check your solution!

Example 5

Solve the following without the use of a calculator:

1 $\sqrt{x + 5} = 8$

$x + 5 = 64$

$\therefore x = 59$

2 $\sqrt{x + 2} - x = 0$

$\sqrt{x + 2} = x$

$x + 2 = x^2$

$x^2 - x - 2 = 0$

$(x + 1)(x - 2) = 0$

$\therefore x = -1$ or $x = 2$

Check: $x = -1$

LHS = $\sqrt{((-1) + 2)} - (-1)$

$= 1 + 1 = 2$

$\neq$ RHS (invalid solution)

Check: $x = 2$

LHS = $\sqrt{(2 + 2)} - 2$

$= 2 - 2 = 0$

= RHS

Therefore, $x = 2$ is the only valid solution.

Summary of the laws of exponents

Laws of exponents	
Multiply powers – add exponents	$a^x \times a^y = a^{x+y}$
Divide powers – subtract exponents	$a^x / a^y = a^{x-y}$
Raise power to a power – multiply exponents	$(a^x)^y = a^{xy}$
Negative exponents can be written as the reciprocal of the power with a positive exponent	$a^{-x} = 1/a^x$
To find the nth root, divide the exponent by n	$\sqrt[n]{a^x} = a^{x/n}$
Anything to the power of zero is 1 (except 0)	$a^0 = 1$

REMEMBER! The bases must be the same!

Questions

Question 1

Simplify the following without using a calculator:

1.1 $3^{4/3}$

1.2 $\sqrt[3]{3^3}$

1.3 $(3^{-3/2})^{2/5}$

1.4 $4^{3/2} - 36^{1/2} + 216^{1/3}$

1.5 $(0{,}0625)^{-3/4}.(0{,}125)^{-4/3}$

1.6 $\sqrt[6]{(64n^{12})^2}$

1.7 $(49m^7n^9)^{6/4}$

1.8 $(81x^3y^7)^{-2/3}.3(x^{-4}y^{-3})^{-2/3}$

1.9 $[\sqrt{169x^3y^4}/(7x^{-3})^{-4}]^{-1}$

1.10 $(4^{2n+3})^{1/7}.(7^{2n-3})^{1/7}/(12^{2n-3})^{1/7}.(5^{2n+3})^{1/7}$

Question 2

Simplify the following without using a calculator:

2.1 $y^{2/3} = 9$

2.2 $k^{1/5} = 3$

2.3 $m^{-4/3} = 0{,}0625$

2.4 $2^{y+3} + 2^y = 9$

2.5 $7^{-k} - 7^{-k-2} = 48$

2.6 $2^{x/2} + 2^{x/2+1} = 24$

2.7 $(7x + 14)(2x - 0{,}0875) = 0$

2.8 $3^{2x} - 2.3^x = 3$

2.9 $16^x + 8.4^x = 48$

2.10 $7^{-x+3} + 7^{2+x} = 392$

Question 3

Simplify the following without using a calculator:

3.1 $\sqrt[3]{5} + 9\sqrt[3]{5} - 4\sqrt[3]{5}$

3.2 $12\sqrt{9} - 3\sqrt{45} + 6\sqrt{72}$

3.3 $(\sqrt{50} - \sqrt{72})/\sqrt{98}$

3.4 $\sqrt[5]{\sqrt{a^{100}}} + \sqrt[3]{\sqrt{a^{15}}} - \sqrt[7]{\sqrt{a^8}}$

3.5 $\sqrt{18} - \sqrt{80} + \sqrt{98}$

3.6 $\sqrt{\sqrt{64} - \sqrt{48}}/\sqrt{50}$

Question 4

Simplify the following without using a calculator:

4.1 $3\sqrt{5} \times 2\sqrt{2}$

4.2 $-4\sqrt{3} \times \left(-5\sqrt{2}\right)$

4.3 $4\sqrt{5}\,(-2\sqrt{2} + 3)$

4.4 $2\sqrt{2}\,(10\sqrt{3} - 8\sqrt{2})$

4.5 $\sqrt[4]{48x^4/y^{24}}$

4.6 $\sqrt[3]{125x^{12}/y^6}$

4.7 $\sqrt[4]{3/243}$

4.8 $\frac{k-2}{k^2}\, if\ k = 1 + \sqrt{3}$

Question 5

Solve for x:

5.1 $\sqrt{(x-1)} = 2$

5.2 $\sqrt{(x+3)} = 5$

5.3 $\sqrt{(x-1)} = 4$

5.4 $x - \sqrt{(-8x-16)} = 5$

5.5 $\sqrt{(x+7)} = -7$

5.6 $\sqrt{11} + \sqrt{y} = \sqrt{(y-2)}$

Equations and inequalities

Overview

In this chapter we will learn about factorisation, and how to complete perfect squares. We'll also have a look at the quadratic formula, and how to solve quadratic inequalities.

This leads to solving simultaneous equations and the nature of roots. Let’s begin!

Solving quadratic equations by factorisation

1.1 Definitions

- A quadratic, or second-degree, equation has the standard form $ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$.
- Its solutions are called roots, and to find the roots we use the zero-product principle:

  If $a \times b = 0$, then $a = 0$ or $b = 0$, or a and b are both 0.

1.2 Solving quadratic equations

- Solving quadratic equations involves the following steps:
  1. Simplify the equation.
  2. Write the equation in standard form.
  3. Factorise the equation.
  4. Apply the zero-product rule.

Example 1

Solve for x:

1 $-x^2 + x = 5 - 2x - 3x^2$

$\therefore 2x^2 + 3x - 5 = 0$

$\therefore (2x + 5)(x - 1) = 0$

$\therefore 2x + 5 = 0 \text{ or } x - 1 = 0$

$\therefore x = -\frac{5}{2} \quad \text{or } x = 1$

2 $(3x + 2)(x - 5) = 0$

$\therefore 3x + 2 = 0 \text{ or } x - 5 = 0$

$\therefore 3x = -2 \quad \text{or } x = 5$

$\therefore x = -\frac{2}{3} \quad \text{or } x = 5$

3 $3x^2 - 5x = 0$

$\therefore x(3x - 5) = 0$

$\therefore x = 0$ or $3x - 5 = 0$

$\therefore 3x = 5$

$\therefore x = \frac{5}{3}$

1.3 Quadratic equations involving fractions

- We solve quadratic equations involving fractions in exactly the same way as we did earlier.
- You have to remember one golden rule here, and that is *the denominator can never be zero*!
- This means that if the denominator contains a variable, then that variable has to be taken into account, e.g. if the denominator is $x - 9$ it means that $x - 9 \neq 0$, therefore $x \neq 9$.

$\frac{1}{x}$ ← The denominator is never = 0.

Example 2

Solve for x:

1 $(x - 3)/(x^2 + 3x + 2) - 4/(-x - 1) = 5/(x^2 - 4)$

$\therefore (x - 3)/(x^2 + 3x + 2) + 4/(x + 1) = 5/(x^2 - 4)$

[Multiply $(-x - 1)$ by -1]

$\therefore (x - 3)/(x + 2)(x + 1) - 4/(x + 1) = 5/(x^2 - 4)$

[Factorise denominator]

$\therefore x \neq -2$ and $x \neq 2$ and $x \neq -1$ [Sort out restrictions]

$\therefore (x - 3)(x - 2) + 4(x + 2)(x - 2) = 5(x + 1)$

[Divide denominator with LCD and multiply]

$\therefore (x - 3)(x - 2) + 4(x^2 - 4) = 5(x + 1)$ [Answer with numerator]

$\therefore x^2 - 5x + 6 + 4x^2 - 16 = 5x + 5$ [Multiply out the brackets]

$\therefore x^2 - 5x + 6 + 4x^2 - 16 - 5x - 5 = 0$ [Take everything to the LHS]

$\therefore 5x^2 - 10x - 15 = 0$ [Simplify to standard form]

$\therefore x^2 - 2x - 3 = 0$

$\therefore (x - 3)(x + 1) = 0$ [Factorise]

$\therefore x - 3 = 0$ or $x + 1 = 0$ [Zero-product principle]

$\therefore x = 3$ or $x = -1$, but $x \neq -1$ [Check for restrictions]

1.4 Quadratic equations involving square roots

- We remove the square root by squaring both sides of the equation.

Example 3

Solve for x:

1 $\sqrt{x - 1} - 1 = -x$

$\therefore \sqrt{x - 1} = -x + 1$ [Isolate the root]

$\therefore (\sqrt{x - 1})^2 = (-x + 1)^2$ [Square both sides]

$\therefore x - 1 = x^2 - 2x + 1$ [Multiply brackets out]

$\therefore x - 1 - x^2 + 2x - 1 = 0$ [Take everything to LHS]

$\therefore -x^2 + 3x - 2 = 0$ [Simplify to standard form]

$\therefore x^2 - 3x + 2 = 0$ [Multiply with a negative]

$\therefore (x - 1)(x - 2) = 0$ [Factorise}

$\therefore x - 1 = 0$ or $x - 2 = 0$ [Zero-product principle]

$\therefore x = 1$ or $x = 2$

1.5 Quadratic equations involving squares

- When squares are involved in quadratic equations, we take the square root on both sides to solve it.
- Remember to make sure that the expression containing the variable is on one side of the equation.

Example 4

Solve for x:

1 $(x-2)^2 = 4$

$\therefore \sqrt{(x-2)^2} = \pm\sqrt{4}$

$\therefore x - 2 = \pm 2$

$\therefore x = 2 \pm 2$

$\therefore x = 0$ or $x = 4$

2 $x^2 = 25$

$\therefore \sqrt{x^2} = \pm\sqrt{25}$

$\therefore x = \pm 5$

1.6 Solving quadratic equations using substitution

- When an equation seems too complicated to work with, look for a common factor in some of the expressions and substitute it with something simple, like the variable *k*.
- Now simplify the equation.
- Then substitute the common factor back in place of *k*, and solve the equation.
- The substitution method makes a complicated equation easier to work with.

Example 5

1 $2(x-6)^2 - 5(x-6) - 12 = 0$

Let $(x-6) = m$

[Remember, you can use any variable here, as long as it is not already in the problem.]

Now we have $2m^2 - 5m - 12 = 0$ $\therefore (2m+3)(m-4) = 0$

$\therefore 2m + 3 = 0$ or $m - 4 = 0$

$\therefore m = \frac{3}{2}$ or $m = 4$, but $m = x - 6$

$\therefore x - 6 = \frac{3}{2}$ or $x - 6 = 4$

$\therefore x = \frac{3}{2} + 6$ or $x = 4 + 6$

$\therefore x = \frac{15}{2}$ or $x = 10$

Completing the square

2.1 Solve for *x* by completing the square

- A perfect square is a rational number (or an expression) that is equal to the square of another rational number (or expression).
- 16 is a perfect square: $4 \times 4 = 4^2 = 16$.
- $(x+5)^2$ is a perfect square: $(x+5)\times(x+5) = (x+5)^2 = x^2 + 10x + 25$.
- If an expression is not a perfect square we can make it one by adding the same term on both sides.

Example 6

Solve the following equations by completing the square.

1 $x^2 - 8x - 6 = 0$ $\qquad x^2 - 8x = 6$

$\therefore x^2 - 8x + (\frac{b}{2})^2 = 6 + (\frac{b}{2})^2$

$\therefore x^2 - 8x + 16 = 6 + 16 = 22$

$\therefore (x-4)^2 = 22$ $\qquad \therefore \sqrt{(x-4)^2} = \pm\sqrt{22}$

$\therefore x - 4 = \pm\sqrt{22}$ $\qquad \therefore x = 4 \pm \sqrt{22}$

$\therefore x = -0{,}6904$ or $x = 8{,}6904$

2 $2x^2 + 8x - 6 = 0$

$x^2 + 4x - 3 = 0$ $\qquad x^2 + 4x = 3$

$\therefore x^2 + 4x + (\frac{b}{2})^2 = 3 + (\frac{b}{2})^2$

$\therefore x^2 + 4x + 4 = 7$

$\therefore (x+2)^2 = 7$

$\therefore \sqrt{(x+2)^2} = \pm\sqrt{7}$

$\therefore x + 2 = \pm\sqrt{7}$

$\therefore x = \pm\sqrt{7} - 2$

$\therefore x = -4{,}6458$ or $x = 0{,}6458$

3.1 Solve for *x* by using the quadratic formula

- The quadratic formula is a general formula that we can derive. It gives us the roots of *any* quadratic equation. We derive it by completing the square:

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

Example 7

Solve for x by using the quadratic formula:

1 $2x^2 + 9x - 6 = 0$ $a = 2;$ $b = 9;$ $c = 6$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{-9 \pm \sqrt{81 - 4(2)(6)}}{2(2)}$$

$$\therefore x = \frac{-9 \pm \sqrt{33}}{4}$$

$\therefore x = -7{,}3723$ or $x = -1{,}6277$

2 $x^2 + 7x = 5$

$\therefore x^2 + 7x - 5 = 0$ $a = 1;$ $b = 7;$ $c = -5$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{-7 \pm \sqrt{49 - 4(1)(-5)}}{2(1)}$$

$\therefore x = \frac{-7 \pm \sqrt{69}}{2}$ $\therefore x = -7{,}6533$ or $x = 0{,}6533$

Quadratic inequalities

4.1 Factorising inequalities

- The only difference between inequalities and equations is the sign, and everything you can do to an equation, you can also do to an inequality.
- But, remember that inequalities have additional rules.
- When dividing by a negative number to solve an inequality, the sign changes.
- Solutions to inequalities are represented on a number line.

Example 8

Solve for x:

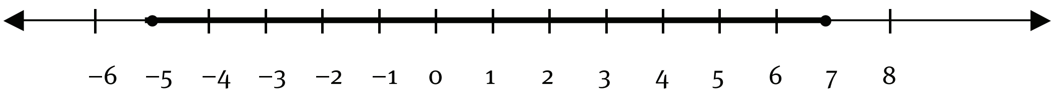
1 $x^2 + 2x - 35 \leq 0$

$\therefore\ (x + 7)(x - 5) \leq 0$ ($x = -7$ and $x = 5$ are the critical values)

For $x + 7$: For $x - 5$:

$x - 7 = 0$	if $x = 7$	$x + 5 = 0$	if $x = -5$
$x - 7 < 0$	if $x < 7$	$x + 5 < 0$	if $x < -5$
$x - 7 > 0$	if $x > 7$	$x + 5 > 0$	if $x > -5$

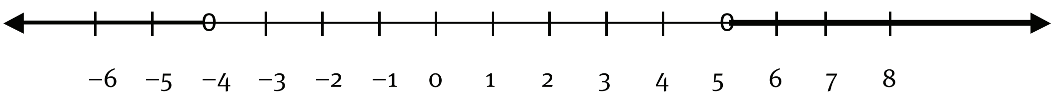
We can represent the result on a number line:



Thus the solution to $(x - 7)(x + 5) \leq 0$ is $-5 \leq x \leq 7$

2 $x^2 - x - 20 > 0$

$(x + 4)(x - 5) > 0$



The solution is $x < -4$ or $x > 5$

5.1 Solve for x and y by solving the equations simultaneously

- When you see two equations with two variables, you can solve for both the variables by solving the equations simultaneously.
- When these equations are drawn on a set of axes, the intersection(s) of the graphs will give you the value(s) of the variables.
- When you draw a parabola (a quadratic equation) and a straight line on the same set of axes, you could have either no intersections, or one intersection or two intersections. This means the variables x and y will have zero, one or two solutions.
- We solve simultaneous equations algebraically as follows:
  1. Write down both equations and number them: (1) for the straight line and (2) for the parabola.
  2. Make y the subject of the straight line equation, number it (3), and substitute (3) into (2).
  3. Use the value for x calculated above (remember, you may find zero, one or two values) and substitute it into (1) to solve for y.

Example 9

Solve for x and y in each case:

1 $3x - y = -9$ (1)

$x^2 + 2x - y = 3$ (2)

From (1): $y = 3x + 9$ (3)

Substitute (3) into (2):

$x^2 + 2x - (3x + 9) = 3$

$x^2 + 2x - 3x - 9 - 3 = 0$

$x^2 - x - 12 = 0$

$(x + 3)(x - 4) = 0$

$x = -3$ or $x = 4$

Substitute the x-values into (3):

For $x = -3$ For $x = 4$

$y = 3(-3) + 9$ $y = 3(4) + 9$

$y = 0$ $y = 21$

For $x = -3,\ y = 0$ For $x = 4,\ y = 21$

$(-3;\ 0)$ $(4;\ 21)$

2 $y - 6x = 12$ (1)

$x^2 + 4x - 9 = 5y$ (2)

From (1): $y = 6x + 12$ (3)

Substitute (3) into (2):

$x^2 + 4x - 9 = 5(6x + 12)$

$x^2 + 4x - 9 = 30x + 60$

$x^2 + 4x - 9 - 30x - 60 = 0$

$x^2 - 26x - 69 = 0$

$x^2 + 26x + 69 = 0$

$x = \dfrac{-b\ \pm\ \sqrt{b^2 - 4ac}}{2a}$

$= -26 \pm \sqrt{676 - 4(1)(69)}/2$

$= (-26 \pm \sqrt{400})/2$

$x = -23$ or $x = -3$

Substitute the x-values into (3):

For $x = -23$ For $x = -3$

$y = 6(-23) + 12$ $y = 6(-3) + 12$

$y = -126$ $y = -6$

For $x = -23;\ y = -126$ For $x = -3;\ y = -6$

$(-23;\ -126)$ $(-3;\ -6)$

6.1 Solving word problems

- When real-life problems that need to be solved mathematically are given in words, the problems are called word problems.
- The four steps to solve word problems:
  1. Understand the problem.
  2. Make a plan (write the problem down in Mathematical terms).
  3. Work your plan (solve the problem).
  4. Assess your answer to the problem (check if your answer holds).

Example 10

Nomsa owns a spaza shop in Pretoria. She buys packets of peanuts for R1 500. She gives 20 of the packets away to loyal customers, and she sells the rest of the packets at R4 more (each) than what she paid for them. She makes a profit of R1 860 on the packets of peanuts. Determine how many packets she bought, the cost price and her selling price.

Step 1: Bought packets of peanuts ⇢ gave 20 away ⇢ sold the balance

Selling price = R4 more than cost price; Profit = R1 860

Step 2: Let x be the number of packets. Then:

Cost price $= 1\ 500/x$ Packets sold $= x - 20$

Total money made $=$ R1 500 + R1 860 = R3 360

Selling price $=$ R3 360$/(x - 20)$; Selling price $=$ cost price + R4

Step 3: $\frac{3\ 360}{x-20} = \left(\frac{1500}{x}\right) + 4$

$3\ 360x = 1\ 500(x - 20) + 4x(x - 20)$

$3\ 360x = 1\ 500x - 30\ 000 + 4x^2 - 80x$

$4x^2 - 1940x - 30\ 000 = 0$

$x^2 - 485x - 7\ 500 = 0$ $a = 1$; $b = -485$; $c = -7\ 500$

$x = (485 \pm \sqrt{(-485)^2 - 4(1)(-7\ 500)})/2(1)$

$x = (485 \pm 515)/2$ $x = -15$ or $x = 500$

Step 4: Since Nomsa cannot buy a negative number of packets, -15 is invalid. Therefore, she bought 500 packets of peanuts.

Cost price $= 1\ 500/500 =$ R3 Selling price $=$ R3 + R4 = R7

Unit 7

The nature of roots

7.1 The roots of an equation

- The roots of an equation are the values of the variables that satisfy that equation, for example

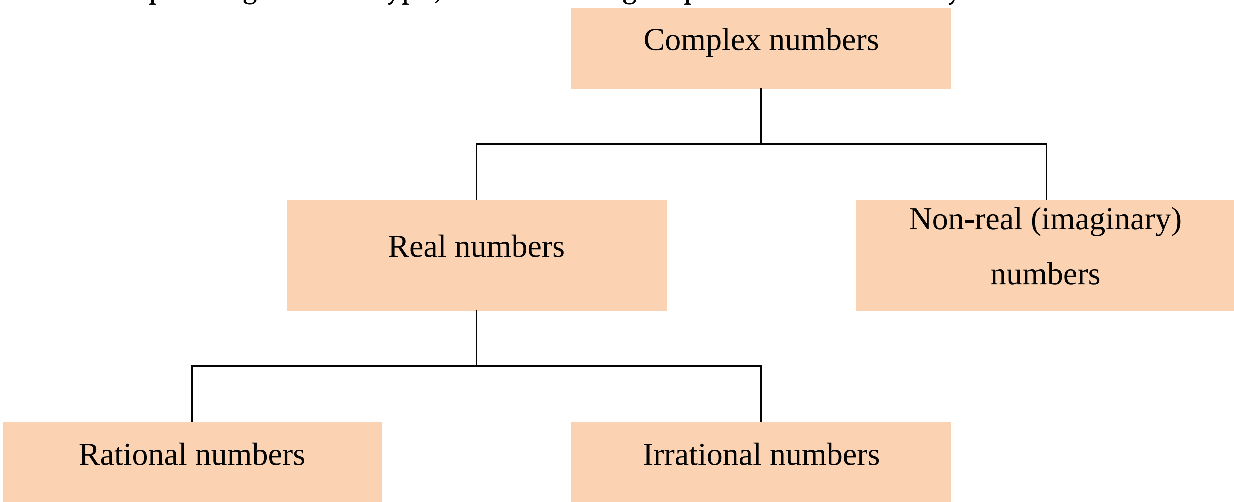
  $x^2 - 4 = 0$

  $(x + 2)(x - 2) = 0$

  Therefore $x = -2$ and $x = 2$ are the roots of the equation.

7.2 Quadratic theory

- Depending on their type, numbers are grouped into a number system:



- When we work out the roots of quadratic equations, we want to find out if the roots are real or non-real numbers, rational or irrational numbers and whether they are equal or unequal.
- We do this by using the value under the square root in the quadratic formula, $b^2 - 4ac$. This value is called the discriminant and is denoted by Δ(delta).
- Use the discriminant to define the roots.

  If:
  - o $\Delta < 0$: the roots are non-real
  - o $\Delta > 0$: the roots are real
  - o $\Delta = 0$: the roots are equal
  - o $\Delta =$ perfect square: the roots are rational
  - o $\Delta \neq$ perfect square: the roots are irrational
- We can use the facts above to prove the nature of any quadratic equation's roots .

7.3 The nature of roots

- By using the formula for Δ as described above, we can find the roots of any quadratic equation.
- By calculating the value of Δ, we can calculate the value of an unknown variable in an equation if the nature of the roots is given, e.g. $kx^2 - 6x + 4 = 0$, where k is the unknown variable.
- Δ also gives us the ability to prove that the nature of the roots of a quadratic equation is of a certain type.

Example 11

1 If $x = 3$ is one of the roots of $x^2 + 3x + k = 0$, determine the value of k and the other root.

Substitute $x = 3$ into the equation to find k:

$3^2 + 3(3) + k = 0$

$\therefore 9 + 9 + k = 0$

$\therefore k = -18$

Now substitute $k = -18$ into the equation and factorise to find the other root.

$x^2 + 3x - 18 = 0$

$(x - 3)(x + 6) = 0$

$\therefore x = 3$ or $x = -6$

$\therefore$ The other root is -6.

2 Find the nature of the roots of:

2.1 $x^2 + 6x - 9 = 0$

$\Delta = b^2 - 4ac$

$= 36 - 4(1)(-9)$

$= 36 + 36$

$= 72$

Δ is positive, but not a perfect square, $\therefore$ roots are real, irrational and unequal.

2.2 $x^2 + 8x + 16 = 0$

$\Delta = b^2 - 4ac$

$= (8)^2 - 4(1)(16)$

$= 64 - 64$

$= 0$

Δ is positive, a perfect square and zero, $\therefore$ roots are real, rational and equal.

3 For which values of m will the equation $mx^2 - 4x + 5 = 0$have :

3.1 equal roots? 3.2 real roots? 3.3 non-real roots?

$mx^2 - 4x + 5 = 0$

$\Delta = b^2 - 4ac$ $= (-4)^2 - 4(m)(5)$

$= 16 - 20m$

3.1 For equal roots, make $\Delta = 0$:

$16 - 20m = 0$

$\therefore 20m = 16$

$\therefore m = \frac{4}{5}$

3.2 For real roots, make $\Delta \geq 0$:

$16 - 20m \geq 0$

$\therefore 16 \geq 20m$

$\therefore m \leq \frac{4}{5}$

3.3 For non-real roots, make $\Delta \leq 0$:

$16 - 20m \leq 0$

$\therefore 16 \leq 20m$

$\therefore m \geq \frac{4}{5}$

4 Prove that the roots of $-2x^2 + (a + b)x + 4 = 0$ are real for all real values of a and b.

$a = -2;$ $b = (a + b);$ $c = 4$

$\Delta = b^2 - 4ac$

$= (a + b)^2 - 4(-2)(4)$

$= (a + b)^2 + 32$

Now we must prove that Δ is greater than 0.

Proof: Any number squared is positive or 0. If we add a positive number to a positive number, the answer must always be positive.

$(a + b)^2 \geq 0$

$32 > 0$

$\therefore (a + b)^2 + 32 > 0$

$\therefore \Delta > 0$; $\therefore$ the roots are real.

Questions

Question 1

Solve the following quadratic equations:

1.1 $(x - 2)(x + 7) = 0$

1.2 $(2y - 3)(y + 5) = 0$

1.3 $x^2 + 21x + 10 = 0$

1.4 $3k(k + 4) = 0$

1.5 $9x^2 - 5x = 0$

1.6 $3k(1 - k) + 5(k + 1) = 0$

1.7 $(3p - 2)(p + 1) + 2 = 0$

1.8 $b(b + 5) = 6$

1.9 $(x - 2)(x + 2) = 6(3x + 5)$

1.10 $4(x - 1)(x + 1) = 3(2 - x) + 5$

Question 2

Solve for x:

2.1 $5/(x - 1) = x/(x + 1)$

2.2 $3/(2x - 6) + x/(x - 3) = 0$

2.3 $(x + 2)/(x - 3) = 7 + 2/(x - 3)$

2.4 $(x - 1)/(x^2 - 9) = 2/(4(x + 3)) - (1 - x)/(x + 1)$

2.5 $1/(x^2 + 2x + 3) + 3/(x^2 + x + 2) = 2/(x^2 + 2x + 3) - 2/(2 - x^2)$

2.6 $\frac{3}{4} + (x + 3)/(2x + 3) = (4x + 3)/(x + 5)$

Question 3

Solve for x:

3.1 $\sqrt{6x + 5} = x$

3.2 $\sqrt{x - 5} = x - 2$

3.3 $\sqrt{2x-3}-x=0$

3.4 $\sqrt{x-6}+x+4=0$

3.5 $2=\sqrt{x^2-27}$

3.6 $\sqrt{x-1}=\sqrt{4x-2}$

Question 4

Solve for x:

4.1 $x^2=81$

4.2 $x^2=27$

4.3 $x^2-16=0$

4.4 $-x^2+49=0$

4.5 $(x+4)^2=48$

4.6 $5(x+5)^2=125$

4.7 $3(x+4)^2-12=0$

4.8 $x^2=(2x-3)^2$

4.9 $3(x-2)^2-16=2$

4.10 $-x=(\frac{1}{2}x+2)^2-6$

Question 5

Solve for x:

5.1 $(x^2-3x)^2-2(x^2-3x)-5=0$

5.2 $(x^2-x)^2=14(x^2-x)+15$

5.3 $\sqrt{x-2}+4=5/\sqrt{x-2}$

5.4 $(\frac{3}{x}+x)^2+\frac{3}{x}-x=19$

5.5 $5(2x^2+x-1)=20(2x^2+x-1)+8$

5.6 $2(x+3)^2-3(x+3)-4=2$

5.7 $24/(3(x-2)=7/(9(x+6))-3$

Question 6

6.1 Solve for p:

$p^2-p-12=0$

6.2 Hence solve for x:

$(x^2+3x)^2-(x^2+3x)-12=0$

Question 7

Given $(a+3)(b+4)=0$, solve for b if $a=-7$.

Question 8

If $(y-2)(x^2+25x-6)=0$, determine y if:

8.1 $x=-7$

8.2 $x=12$

Question 9

Solve for x by completing the square:

9.1 $x^2+2x+4=0$

9.2 $x^2-5x+15=0$

9.3 $x^2-x+20=0$

9.4 $x^2+6x=-5$

9.5 $13x=10+x^2$

9.6 $x^2-6x=3$

Question 10

Solve for y:

10.1 $3y(y-3)-3=0$

10.2 $8y^2-2y+5=0$

10.3 $-y^2-3y=-2$

10.4 $6=3y(y+4)$

10.5 $\frac{1}{8}y^2-2y+5=0$

10.6 $(y+4)^2+(y-4)(y+3)=2$

10.7 $(y-3)(4y+2)-9=0$

10.8 $y(y+1{,}5)+3y=3y(y-9)+0{,}3$

10.9 $5(y+3)=3y^2-5$

10.10 $2/(y(y+2))=3/(y+2)-y/(y+1)$

Question 11

Solve for x:

11.1 $x^2+7x-14>0$

11.2 $2x^2-9x-17<0$

11.3 $x^2>3x-7$

11.4 $-5x\geq -3x^2+8$

11.5 $x^2\leq 6x-9$

11.6 $3x+2<5x-6x^2$

Question 12

Solve for each of the variables in the following equations:

12.1 $9x - y = 12$
$13y^2 + 4xy - x = 0$

12.2 $y + 6 - 13x = 0$
$xy = -13$

12.3 $y + x - 5 = 0$
$x^2 - 4 = y - 6x$

12.4 $x + y = 5$
$3x^2 + 5x - 19 = 0$

12.5 $3a + 5b = 17a - 6$
$(2a - 4b)(a + b) = 0$

12.6 $n + m = -13$
$n = 3m^2 + 4m - 6$

Question 13

13.1 The sum of two numbers is 15, while the product of the same numbers is 36. Find the two numbers.

13.2 Two fruit pickers are picking fruit. Fruit picker A picks 750 pieces of fruit in 4 hours, while fruit picker B picks 750 pieces of fruit in 3 hours. How long will it take them to pick 750 pieces of fruit if they work together?

13.3 Two trains travel 1 600 km from Cape Town to Pretoria. Train A is 15 km per hour faster than train B, and arrives at Pretoria Station 2 hours ahead of train B. Determine the speed of train B.

13.4 Nomsa and Emily own a home cleaning business. Nomsa takes 2,5 hours longer than Emily to clean a home. If they work together, they take 6 hours to clean a home. How long will it take each of them to clean a home on their own?

Question 14

14.1 Given that -6 is one of the roots of the equation $x^2 + mx - 30 = 0$

14.1.1 Determine the value of m.

14.1.2 Now determine the other root of the equation.

14.2 If $-\frac{7}{2}$ is one of the roots of $x^2 + kx - 7 = 0$, determine the value of k and the other root.

14.3 Determine the value of p that will make the following expression a perfect square:
$x^2 - 10x + p$

14.4 Determine the value of n that will make this equation a perfect square:
$3x^2 - 3x + n$

Question 15

Find the value of the discriminant and describe the nature of the roots without solving the equation.

15.1 $3x^2 - 6x + 4 = 0$

15.2 $x^2 + 64 = 12x$

15.3 $3x = 5x^2 - 6$

15.4 $x - 3 = -12x^2$

Question 16

For which values of k will $3x^2 - 3x + k$ have equal roots?

Question 17

Find the values of r for which $x^2 - 3rx + r = 0$ has real roots.

Question 18

Prove that $x^2 + (k - 1)x + k = 0$ has rational roots for all rational values of k.

Question 19

Show that the roots of $x^2 + m = (m + 2)x$ will be real for all real and unequal values of m.

Chapter 3

Number patterns

Overview

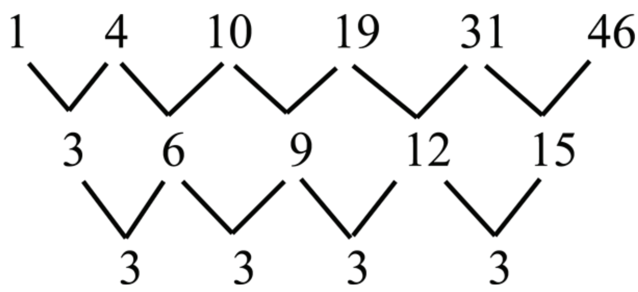
	Unit 1 Page 32	
Chapter 3 Page 31 Number patterns	Number patterns with a constant second difference	• Finding the constant second difference

So far in your Mathematics career, you have learnt about linear number patterns. This means that there is a constant difference between consecutive terms in the sequence.

In this chapter we learn about number patterns with a constant *second* difference. This means that there is a constant difference between the first difference values in a sequence of numbers.

Remember, we use T_n to represent a general term in a sequence, where $n \in N$.

1.1 Finding the constant second difference



- When you have a sequence, say : $1;\quad 4;\quad 10;\quad 19;\quad 31;\quad 46;$ it is easy to work out the first sequence of differences. They are $3;\quad 6;\quad 9;\quad 12;\quad 15.$
- Now, to work out the second differences, we do exactly the same as we did to calculate the first differences, but we use the sequence of first differences instead of the original sequence of numbers.
- The second differences are $3;\quad 3;\quad 3;\quad 3.$

  And yes, this sequence has a constant second difference! If the second differences of a sequence are constant, then the formula for the general term is quadratic. That means the general term will be of the form $T_n = an^2 + bn + c$.

Example 1

1 Find the general term for the sequence $-1;\ 9;\ 23;\ 41;\ \dots$

The first differences are $10;\ 14;\ 18;\ \dots$

The second differences are $4;\ 4;\ \dots$

$\therefore$ The form of the general term will be $T_n = an^2 + bn + c$.

Let's start off by substituting T_1 and T_2.

$T_1 = -1 \qquad \therefore -1 = a + b + c \qquad (1)$

$T_2 = 9 \qquad \therefore\ 9 = 4a + 2b + c \qquad (2)$

Then (1) − (2) yields $-10 = -3a - b \qquad (3)$

Substituting $T_3 = 23$ we obtain

$23 = 9a + 3b + c \qquad (4)$

Then (4) – (2) yields $14 = 5a + b \qquad (5)$

and (5) + (3) yields $4 = 2a \qquad \therefore a = 2$ and $b = 4$

Substituting $a = 2$ and $b = 4$ into (1) we obtain

$-1 = 2 + 4 + c \qquad \therefore\ c = -7 \qquad \therefore\ T_n = 2n^2 + 4n$

2 Find the general term for the sequence $0;\ 30;\ 92;\ 186;\ \dots$

The first differences are $30;\ 62;\ 94;\ \dots$

The second differences are $32;\ 32;\ \dots$

∴ The form of the general term will be $T_n = an^2 + bn + c$.

Substitute T_1 and T_2.

$T_1 = 0$ ∴ $0 = a + b + c$ (1)

$T_2 = 30$ ∴ $30 = 4a + 2b + c$ (2)

(1) – (2) yields $-30 = -3a - b$ (3)

Substituting $T_3 = 92$ we obtain

$92 = 9a + 3b + c$ (4)

(4) – (2) yields $62 = 5a + b$ (5)

and (5) + (3) yields $32 = 2a$

∴ $a = 16$ and $b = -18$

Substituting $a = 16$ and $b = -18$ into (1) we obtain

$0 = 16 - 18 + c$

∴ $c = 2$

∴ $T_n = 16n^2 - 18n + 2$

Questions

Question 1

Consider the following sequences:

1.1 $9;\ 16;\ 25;\ 36;\ \dots$

1.2 $4;\ 15;\ 32;\ 55;\ \dots$

1.3 $1;\ 16;\ 33;\ 52;\ \dots$

1.4 $-4;\ 5;\ 24;\ 53;\ \dots$

1.5 $5;\ 33;\ 95;\ 191;\ \dots$

For each sequence:

i Determine the second differences between the terms.

ii Write down the next five terms of each sequence.

iii Determine the general term of the sequence.

iv Determine the value of T_{12} and T_{30}.

Question 2

The general term for a sequence with a quadratic pattern is given by

$T_1 = 1;\ T_n\ =\ x^2 - x + 1\ for\ n \geq 1.$

2.1 Write down the first seven terms of the sequence.

2.2 Express the general term in the form $T_n = an^2 + bn + c$.

2.3 Which term in the sequence is 133? Which term is 1 407?

Analytical geometry

Overview

Chapter 4 Page 35 Analytical Geometry	Unit 1 Page 36 The inclination of a line	• Finding the gradient and inclination of a straight line
	Unit 2 Page 38	
	The equation of a straight line	• The gradient and the y-intercept • The gradient and one point on the line • Equation of a line going through two points • Equation of a line through one point and parallel or perpendicular to a given line
	Useful Information Page 39	

Analytical Geometry is also known as Coordinate Geometry and combines Geometry and Algebra. In this chapter you will learn how to calculate the inclination of a line, how to find the equation of a straight line graph if certain coordinates and other information is given, and how to solve problems involving triangles.

The inclination of a line

1.1 Finding the gradient and inclination of a straight line

- The gradient of a line is its slope or steepness.
- The following formulae are useful to calculate different values:
  - The distance between points A and B:
    AB = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
  - The gradient of a line:
    $m = (y_2 - y_1)/(x_2 - x_1)$
  - The midpoint between two points:
    Midpoint = $(\frac{(x_1+x_2)}{2}; \frac{(y_1+y_2)}{2})$
- The inclination of a line AB is the angle θ that is formed between the line and the positive x-axis.
- For acute angles $(0 < \theta \leq 90°)$ the gradient is positive and $\tan\theta$ is positive.
- For obtuse angles $(0 < \theta \leq 180°)$ the gradient is negative and $\tan\theta$ is negative.
- The inclination of AB = θ where $\tan\theta$ = gradient of AB (we denote this as m_{AB}).
- Two parallel lines with inclinations θ and α, have $m_1 = m_2$ and $\tan\theta = \tan\alpha$.
- Two perpendicular lines with inclinations θ and α have $m_1 \times m_2 = -1$ and $\tan\theta \times \tan\alpha = -1$.

Example 1

1 Determine the inclination of the line with gradient -2.

$\tan\theta = m$

$\therefore \tan\theta = -2$

$\therefore \theta = -63{,}43°$

2 Determine the inclination of line KY if K$(-12;\ 9)$ and Y$(6;\ 3)$.

$\tan\theta = m_{KY}$

$= (y_2 - y_1)/(x_2 - x_1)$

$= (3 - 9)/(-12 - 6)$ $= \frac{-6}{-18} = \frac{1}{3}$

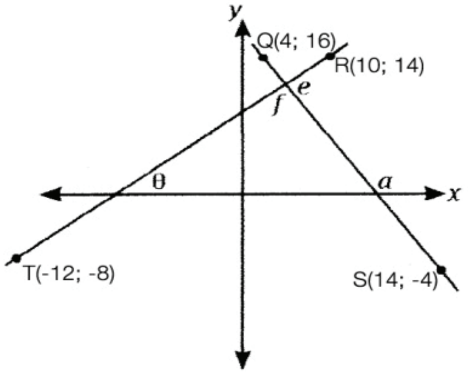
$\therefore \theta = 18{,}43°$

3 Calculate the gradient of the line with inclination $52{,}6°$.

$m = \tan\theta$ $\therefore m = \tan 52{,}6°$

$= 1{,}31$

4 Consider the sketch below and calculate the sizes of θ, α, e and f.



5 Determine whether LM is parallel or perpendicular to YZ in each case.

5.1 L(7; 6), M(−5; −1), Y(7; 2), Z(−5; −5)

5.2 L(−2; 5), M(−8; 2), Y(4; 3), Z(−2; 0)

5.3 L(6; 1), M(−1; −3), Y(6; 4), Z(−1; 0)

5.4 L(−6; 3), M(−3; −1), Y(7; 7), Z(3; 4)

5.5 L(−1; −6), M(3; −1), Y(7; −1), Z(2; 3)

The equation of a straight line

2.1 The gradient and the *y*-intercept

- When we are given the gradient and the y-intercept of a straight line, we use the equation $y = mx + c$ to find the equation with the given information. Remember that m is the gradient and c is the y-intercept.

2.2 The gradient and one point on the line

- When we are given the gradient and one point on the straight line, we use the equation $y - y_1 = m(x - x_1)$ to find the equation of the straight line with the given information.

2.3 Equation of a line going through two points

- When we are given two points on a line, we first have to calculate the gradient m and then use the same equation as in 2.2.

2.4 Equation of a line through one point and parallel or perpendicular to a given line

- To find the equation of a line such as the one described above, we need to follow the following simple steps:
  1. Write the equation of the given line in standard form to find m and c.
  2. Calculate the gradient of the line with the unknown equation by using the rules about parallel and perpendicular lines as discussed in Chapter 4, Unit 2 above.
  3. Substitute m and the coordinates of the given point into the standard equation for a straight line to find the equation.

Example 2

1 Determine the equation of the line with gradient -2 and y-intercept 17.

$y = mx + c$, therefore $y = -2x + 17$.

2 Determine the equation of the line with gradient 7 and that goes through point $(-2, 7)$.

$y - y_1 = m(x - x_1)$

$\therefore\ y - 7 = 7(x + 2)$

$\therefore\ y = 7x + 14 + 7$

$\therefore\ y = 7x + 21$

3 Find the equation of the line that goes through the points K$(3;\ -2)$ and L$(-4;\ -3)$.

First, we find the gradient:

$m_{\text{KL}} = (y_2 - y_1)/(x_2 - x_1)$

$= (-3 - (-2))/(-4 - 3)$

$= -1/-7$

$= \frac{1}{7}$

Second, we select any one of the two points. Let's use L$(-4;\ -3)$.

$y\ - y_1 = m(x - x_1)$

$\therefore\ y - (-3) = \frac{1}{7}(x - (-4))$

$\therefore\ y = \frac{1}{7}\ x + \frac{4}{7} - 3$

$= \frac{1}{7}x - \frac{17}{7}$

Useful information

The median of a triangle

- The median of a triangle bisects both the opposite side and the area of a triangle.
- If KL is the median of a triangle, we can calculate the coordinates of L by using the midpoint formula.
- We can use points K and L to find the equation of the median line.

The altitude of a triangle

- The altitude of a triangle is perpendicular to the opposite side.
- If we want to find the altitude PQ, we have to find the gradient of line ST that is perpendicular to it.
  We know $m_{\text{ST}} \times m_{\text{PQ}} = -1$, so we can calculate m_{PQ}. Now, we can use m_{PQ} and the coordinates of point P to find the equation of altitude PQ.

Perpendicular bisectors

- Two lines are perpendicular bisectors if they cross each other at an angle of 90° (perpendicular) and divide each other exactly in half (bisectors).
- To calculate the equations of two perpendicular bisectors:
  1. Find the coordinates of the point at the intersection using the midpoint formula.
  2. Calculate the gradient of one of the lines (let's call it line 1).
  3. Now, use the $m_1 \times m_2 = -1$ property to calculate m_2.
  4. Use the coordinates of the point of intersection and m_2 to find the equation of line 2.

Questions

Question 1

Determine the inclination of the lines given either the gradient or two points:

1.1 15

1.2 12/2

1.3 ¾

1.4 −3

1.5 −0,125

1.6 Q(−4; −7) and R(−6; 2)

1.7 Q(8; −3) and R(5/7; 13)

1.8 Q(−3; −1) and R(1; −4)

1.9 Q(−2; −4) and R(−6; −1)

1.10 Q(5; −1) and R(−12; 7)

Question 2

Calculate the gradient of the line with inclination:

2.1 72°

2.2 250°

2.3 13°

2.4 −126°

2.5 −50,8°

2.6 185,15°

Question 3

Determine whether AB is parallel, perpendicular or neither, to CD in each case.

3.1 A(−1; 5), B(7; 8), C(5; −2), D(16; 3)

3.2 A(−5; −5), B(7; 2), C(−5; −1), D(7; 6)

3.3 A(−9; 6), B(−5; −7), C(9; −3), D(4; 7)

3.4 A(2; 5), B(3; 2), C(−9; 3), D(1; 3)

3.5 A(9; 1), B(8; 4), C(−4; 3), D(2; 5)

Question 4

Determine the equation of the line:

4.1 with gradient = 5, y-intercept = −17

4.2 with gradient = −2/5, y-intercept = 9

4.3 with gradient m = 7/3, passing through point (−13; 5)

4.4 with gradient m = −5, passing through point (−2; −1)

4.5 going through points (−4; 5) and (−1; −1)

4.6 going through points (13; −12) and (−5; 9)

4.7 BC if BC is parallel to 5y = x − 15 and passes through the point (−1; 4)

4.8 KT if KT is perpendicular to line BS, which has equation 2y = 6x − 7 and passes through point (4/3; 0,5)

Overview

Chapter 5 Page 42 Functions	**Unit 1 Page 43** Investigating the effect of the parameter *p*	• Point by point plotting • Draw sketch graphs • Determining the equations of graphs • Interpreting graphs
	Unit 2 Page 46 Average gradient between two points on a curve	• Average gradient
	Unit 3 Page 47 Trigonometric graphs	• Revision of the basic graphs • Investigating the effect of the parameter *p* on the graph of $y = \sin(x + p)$ • Sketch graphs of trigonometric functions containing at most two of the parameters *a*, *p* and *q* • Interpreting graphs

Functions are all around us, and can be used to describe the relationship between things mathematically. In this chapter you revised key concepts from Grade 10. You will also investigate the effect of different parameters on certain functions and how to calculate the average gradient between two points. You will work with trigonometric graphs, and learn how different parameters influence the shape and position of them.

1.1 Point by point plotting

- When we want to plot functions, it is easy to remember what happens to the shape and position of the graphs when variables are introduced.
- Let's look at a **parabola** $y = ax^2$. If x is substituted with $x + p$, then the value of p will have the following influence on the position of the graph:
  - If $p > 0$, the graph will shift p units to the left.
  - If $p < 0$, the graph will shift p units to the right.

  The shape does not change, which means the graphs are congruent.
- Now let's consider the **hyperbola** $y = a/(x + p)$:
  - If $p > 0$, the graph shifts p units to the left (remember the asymptote also shifts p units to the left).
  - If $p < 0$, the graph shifts p units to the right (remember the asymptote also shifts p units to the right).

  The shape does not change, which means the graphs are congruent.
- Lastly we have to consider the **exponential function** $y = bx + p + q$:
  - If $p > 0$, the graph shifts p units to the left (the asymptote stays the same).
  - If $p < 0$, the graph shifts p units to the right (the asymptote stays the same).

  The shape does not change, which means the graphs are congruent.

1.2 Draw sketch graphs

- Remember that, unless stated otherwise, x and y are both elements of the set of Real numbers (R) and the range is determined by the y-value of the turning point.
- If x and y are elements of the set of Integers (Z) then we plot individual points because the data is discrete.
- To sketch the graph of the **parabola** ($\boldsymbol{y = a(x + p)^2 + q}$), we need information about:
  - the shape
    - If $a > 0$, the graph is concave (arms upwards).
    - If $a < 0$, the graph is convex (arms downwards).
  - the axis of symmetry ($x + p = 0 \therefore x = -p$)
  - the turning point
  - the y-intercept
  - the x-intercepts(s)
  - q causes vertical shifts:
    - If $q > 0$, the graph shifts q units upwards.
    - If $q < 0$, the graph shifts q units downwards.

▪

- We can simplify each equation of the form $y = a(x+p)^2 + q$ *to* $y = ax^2 + bx + c$.
  The coordinates of the turning point will be $(-b/2a;\ (4ac - b^2)/4a)$.
- To sketch the graph of the **hyperbola** $(\boldsymbol{y = a/x + p + q})$, we need
  - the horizontal asymptote: $y = q$
  - the vertical asymptote: $x + p = 0 \ \therefore\ x = -p$
  - the y −intercept: let $x = 0$
  - the x-intercept: let $y = 0$.
- If nothing is stated about the domain of the graph it is always $x \in R$.
  Similarly $y \in R$, and you must state the restrictions. The domain and range are governed by the asymptotes, and these values need to be excluded from them.
- To sketch the graph of the **exponential function** $(\boldsymbol{y = a.bx + p + q,\ \ b \neq 1,\ \ b > 0})$, we need
  - the shape: $b > 1$ or $0 < b < 1$
  - the horizontal asymptote: $y = q$
  - the y-intercept: let $x = 0$
  - the x-intercept (only after a vertical shift and if asked for): let $y = 0$
  - the range: determined by the equation of the horizontal asymptote.

1.3 Determining the equations of graphs

- When you are asked to determine the equation of a given graph, certain information will be given that will allow you to easily get to the equation.
- To find the equation of a **parabola**, there are two scenarios that you need to consider:
  - if the turning point and one other point are given, you use the form $f(x) = a(x + p)^2 + q$
  - if the x-intercepts and one other point are given, you use the form $y = a(x - x_1)(x - x_2)$
- Remember, if the graph passes through the origin, then (0; 0) lies on the graph!
- To find the equation of a **hyperbola**, you only need the horizontal and vertical asymptotes and one point on the graph.
- p and q can be substituted immediately as they relate to the equations of the asymptotes.
- To find the equation of the **exponential function**, you will need the equation of the horizontal asymptote and one point on the graph, and they will be given.
- p and q can be substituted immediately as they relate to the equations of the asymptotes.

1.4 Interpreting graphs

- There is a lot of information that can be gleaned from a graph that has been sketched.
- Remember, an ordered number pair indicates the position of a point on the Cartesian plane, as well as the distance of the point from the axes.
  B(-3; 5) is 3 units from the x-axis and 5 units from the y-axis. Distance is never negative!

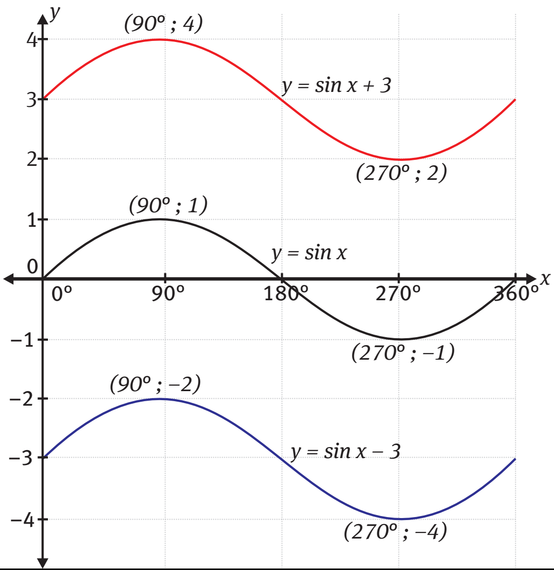
Average gradient between two points on a curve

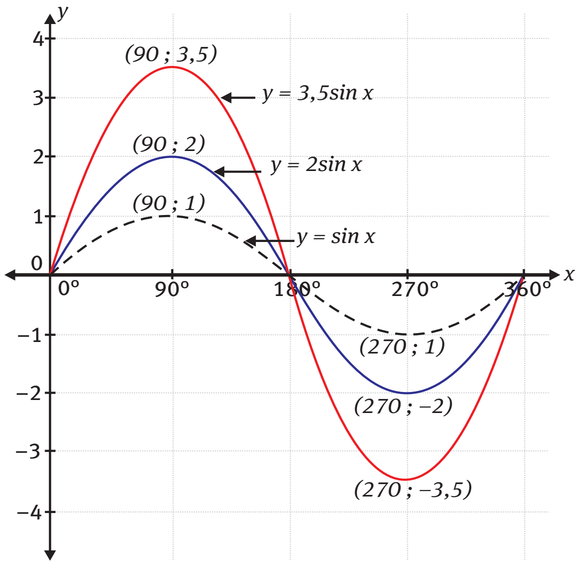
2.1 Average gradient

- The average gradient of a curve between any two points is equal to the gradient of the straight line joining the two points.
- We use the formula $m = (y_2 - y_1)/(x_2 - x_1)$ to calculate the gradient of the straight line joining two points on a curve.

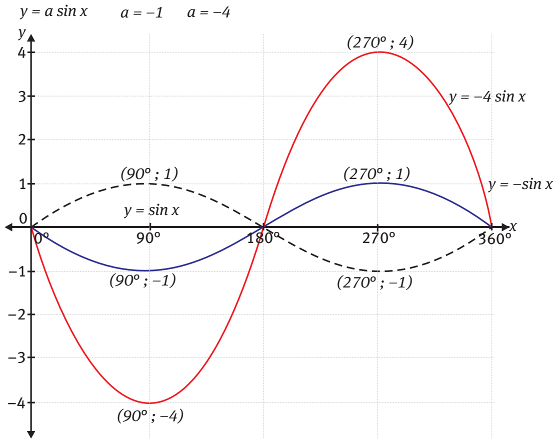
3.1 Revision of the basic graphs

The effect of a and q on the graph $y = a \sin x + q$ for $x \in [-360°;\ 360°]$



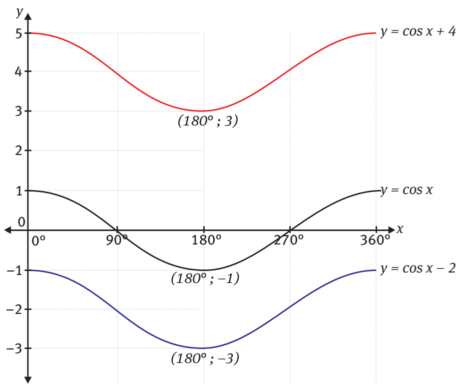


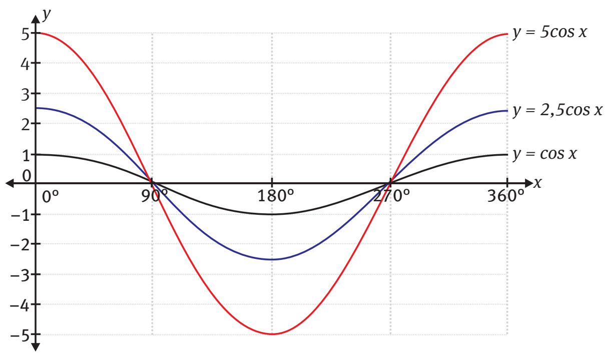
Manipulating the sin graph, a > 0	
$q > 0$: the graph shifts upwards. $q < 0$: the graph shifts downwards. The shape does not change.	$a > 0$: multiplying by a causes a vertical stretch of the basic graph. The shape of the basic graph changes.
Domain: $x \in [-360°;\ 360°]$	**Domain:** $x \in [-360°;\ 360°]$
Range: $y = \sin x + 3$: $2 \leq y \leq 4$; $[2;\ 4]$ $y = \sin x$: $-1 \leq y \leq 1$; $[-1;\ 1]$ $y = \sin x - 3$: $-4 \leq y \leq -2$; $[-4;\ -2]$	**Range:** $y = 3{,}5 \sin x$: $-3{,}5 \leq y \leq 3{,}5$; $y \in [-3{,}5;\ 3{,}5]$ $y = 2 \sin x$: $-2 \leq y \leq 2$; $y \in [-2;\ 2]$ $y = \sin x$: $-1 \leq y \leq 1$; $y \in [-1;\ 1]$
Amplitude: ((highest y –value) – (lowest y-value))/2 For all three graphs this equals 1.	**Amplitude:** The amplitude varies for the three graphs, e.g. for $y = 3{,}5 \sin x$: $(3{,}5 - (-3{,}5))/2 = 3{,}5$
Period: 360° for all three graphs	**Period:** 360° for all three graphs



Manipulating the sin graph, a < 0
$a < 0$: multiplying by negative a causes a reflection of $y = a \sin x$ in the x −axis
Domain: $x \in [-360°;\ 360°]$
Range: $y = (-1)\sin x$: $-1 \leq y \leq 1$; $y \in [-1;\ 1]$ $y = (-4)\sin x$: $-4 \leq y \leq 4$; $y \in [-4;\ 4]$
Amplitude: The amplitude varies for the three graphs, e.g. for $y = (-4)\sin x$: $(4 - (-4))/2 = 4$
Period: 360° for all three graphs

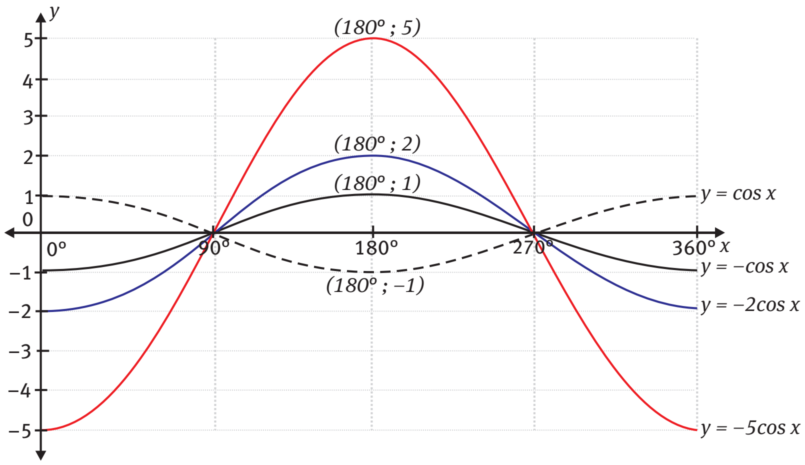
The effect of a and q on the graph $y = a \cos x + q$ for $x \in [-360°;\ 360°]$





Manipulating the cos graph, a > 0	
$q > 0$: the graph shifts upwards. $q < 0$: the graph shifts downwards. The shape does not change.	$a > 0$: multiplying by a causes a vertical stretch of the basic graph.

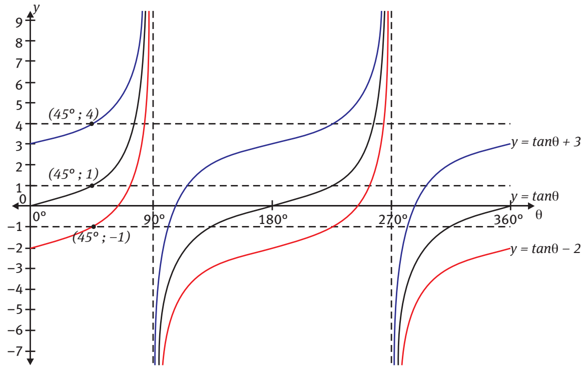
Domain: $x \in [-360°; 360°]$	**Domain:** $x \in [-360°; 360°]$
Range: $y = \cos x + 4$: $3 \le y \le 5$; $[3; 5]$ $y = \cos x$: $-1 \le y \le 1$; $[-1; 1]$ $y = \cos x - 2$: $-3 \le y \le -1$; $[-3; -1]$	**Range:** $y = 5\cos x$: $-5 \le y \le$; 5 $y \in [-5; 5]$ $y = 2\cos x$: $-2{,}5 \le y \le 2{,}5$; $y \in [-2{,}5; 2{,}5]$ $y = \cos x$: $-1 \le y \le 1$; $y \in [-1; 1]$
Amplitude: ((highest y –value) – (lowest y-value))/2 For all three graphs this equals 1.	**Amplitude:** The amplitude varies for the three graphs, e.g. for $y = 5\ cos\ x$: $(5 - (-5))/2 = 5$
Period: 360° for all three graphs	**Period:** 360° for all three graphs.

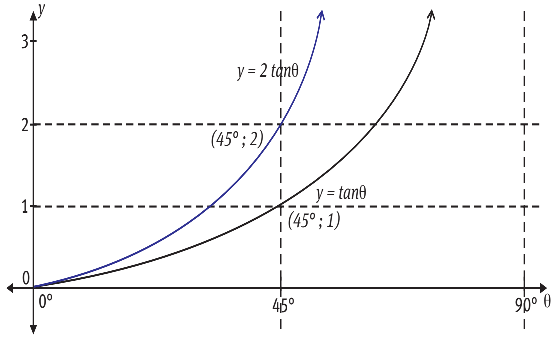


Manipulating the cos graph, a < 0
$a < 0$: multiplying by negative a causes a reflection of $y = a \cos x$ in the x-axis.
Domain: $x \in [-360°; 360°]$
Range: $y = (-5)\cos x$: $-5 \le y \le 5$; $y \in [-5; 5]$ $y = -\cos x$: $-1 \le y \le 1$; $y \in [-1; 1]$
Amplitude: The amplitude varies for the three graphs, e.g. for $y = (-5)\cos x$: $(5 - (-5))/2 = 5$
Period: 360° for all three graphs.

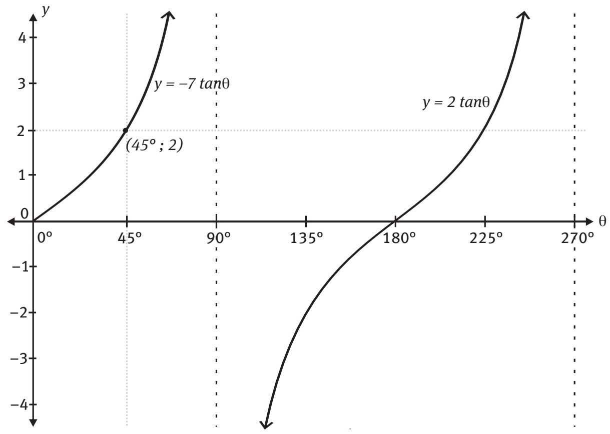
The effect of a and q on the graph $y = a\ tan\ \theta + q$ for $x \in [-360°; 360°]$

Note: The horizontal axis is no longer the x –axis, but is now the θ –axis!





Manipulating the tan graph	
Domain: $x \in [-360°; 360°]$	**Domain:** $x \in [-90°; 90°]$
Range: Undefined	**Range:** Undefined
Period: 180°	**Period:** 180°
Amplitude: Undefined	**Amplitude:** Undefined
Asymptotes: -270°, -90°, 90° and 270°	**Asymptotes:** -90° and 90°



Manipulating the tan graph
$y = \tan\theta$
Domain: $x \in [-90°; 270°]$
Range: Undefined
Period: 180°

Amplitude: Undefined
Asymptotes: -90°, 90° and 270°

3.2 Investigating the effect of the parameter *p* on the graph of $y = \sin(x + p)$

- The parameter p causes a horizontal shift of the sine graph:
  - If $p > 0°$, the graph shifts p degrees to the left.
  - If $p < 0°$, the graph shifts p degrees to the right.
  - The period, amplitude and shape stay the same.
- The parameter p causes a horizontal shift of the cosine graph:
  - If $p > 0°$, the graph shifts p degrees to the left.
  - If $p < 0°$, the graph shifts p degrees to the right.
  - The period, amplitude and shape stay the same.
- The parameter p causes a horizontal shift of the tan graph:
  - If $p > 0°$, the graph shifts p degrees to the left.
  - If $p < 0°$, the graph shifts p degrees to the right.
  - The asymptotes shift to the left or right by $p°$.
  - The period stays the same, and the amplitude is undefined.

3.3 Sketch graphs of trigonometric functions containing at most two of the parameters *a*, *p* and *q*

- Now that we know what the parameters mean and what influence they have on the trigonometric graphs, it is easy to sketch the graphs of these functions.
- Remember to sketch the basic function of the graph you are trying to sketch in light pencil to guide you.
- The basic graphs that we have studied up to now are:

Basic trigonometric graphs			
Basic functions	$y = \sin x$	$y = \cos x$	$y = \tan x$
q causes a vertical shift	$y = \sin x + q$	$y = \cos x + q$	$y = \tan x + q$
a causes a vertical shift	$y = a \sin x$	$y = a \cos x$	$y = a \tan x$

k influences the period	$y = \sin kx$	$y = \cos kx$	$y = \tan kx$
p causes a horizontal shift of $p°$ to the left ($p > 0$) or to the right ($p < 0$)	$y = \sin(x + p)$	$y = \cos(x + p)$	$y = \tan(x + p)$

- When you are asked to sketch a graph with more than one parameter influencing it, don't fret. Take it one step at a time:
  - Start on the LHS – is there a value for a? What does this mean?
  - Move on – is there a value for p? What does this mean?
  - Move on – is there a value for q? What does this mean?

  Once you have answers to all these questions you will be able to sketch the graph.

3.4 Interpreting graphs

- Remember: Every ordered number pair on the graph of a trigonometric function indicates the size of the angle and the value of the trigonometric ratio for that specific angle.
- Each y-value indicates the distance from that point to the x −axis. Distance is never negative!

Questions

Question 1

1.1 Draw sketch graphs of the group of functions given below on the same set of axes:

$f(x) = x^2$ $\quad$ $g(x) = -(x - 3)^2$

$h(x) = -(x + 2,5)^2$ $\quad$ $i(x) = (x - 4)^2$

1.2 Write down the equation of the axis of symmetry of each graph.

1.3 How do the equations of the axes of symmetry correspond to the values of p in the equations of the functions $g(x)$, $h(x)$ and $i(x)$?

Question 2

Draw sketch graphs of the following functions.

For each graph, clearly indicate the axis of symmetry, the coordinates of the intercepts with the axes as well as those of the turning points.

2.1 $y = \frac{1}{3}x^2 - 9$

2.2 $y = \frac{1}{3}(x - 3)^2$

2.3 $y = \frac{1}{3}(x + 7)^2$

2.4 $g(x) = -12x^2 + 30$

2.5 $f(x) = -12(x - 7)^2$

2.6 $h(x) = -12(x + 1)^2$

2.7 $p(x) = 7(x + 3)^2 - 2$

2.8 $k(x) = 7(x - 3)^2 + 2$

2.9 $g(x) = -5(x - 4)^2 - 3$

2.10 $g(x) = -5(x + 3)^2 + 6$

Question 3

Given $h(x) = x^2 - 5$.

3.1 Calculate:

3.1.1 $h(3)$

3.1.2 $h(x - 3)$

3.1.3 $h(x) - 3$

3.1.4 $h(x - 3) + 7$

3.2 Explain the difference between the graphs of $h(x)$, $h(x - 3)$ and $h(x) - 3$.

3.3 Determine an expression for $h(2/x) + h(x + 2) - 5$.

Question 4

Convert the equations of the form $y = ax^2 + bx + c$ given below to equations of the form $y = a(x + p)^2 + q$.

Draw sketch graphs of each function and clearly indicate the coordinates of the intercepts with the axes, as well as the turning point of each graph. Write down the domain and range of each function.

4.1 $y = x^2 + 10x + 2$

4.2 $y = x^2 + 15x + 56$

4.3 $f(x) = 2x^2 + 7x - 9$

4.4 $p(x) = -4x^2 + 15x + 3$

Question 5

Draw sketch graphs of each of the following functions. Indicate the coordinates of the intercepts with the axes, as well as the turning points.

Write down the domain and range of each function.

5.1 $f(x) = \frac{1}{3}x^2 + 3x - 8$

5.2 $g(x) = -\frac{1}{4}x^2 - \frac{1}{4}x - 28$

5.3 $y = x^2 + 9x - 7$

5.4 $y = x^2 - 3x - 14$

Question 6

Given the function $f(x) = x^2 - 7$, determine the equation of f after each of the following shifts. Leave your answers in the form $y = \ldots$

The graph of f is shifted:

6.1 3 units upwards

6.2 9 units downwards

6.3 4 units to the left

6.4 7 units to the right

6.5 1/3 of a unit upwards, and 3/4 of a unit left

6.6 5 units downwards, and 3,5 units right

Question 7

Draw sketch graphs of the functions given. On each graph, indicate the coordinates of the intercepts with the axes as well as the equations of the asymptotes.

7.1 $y = 2/(x - 3) + 7$

7.2 $y = -2/(x + 2) + 3$

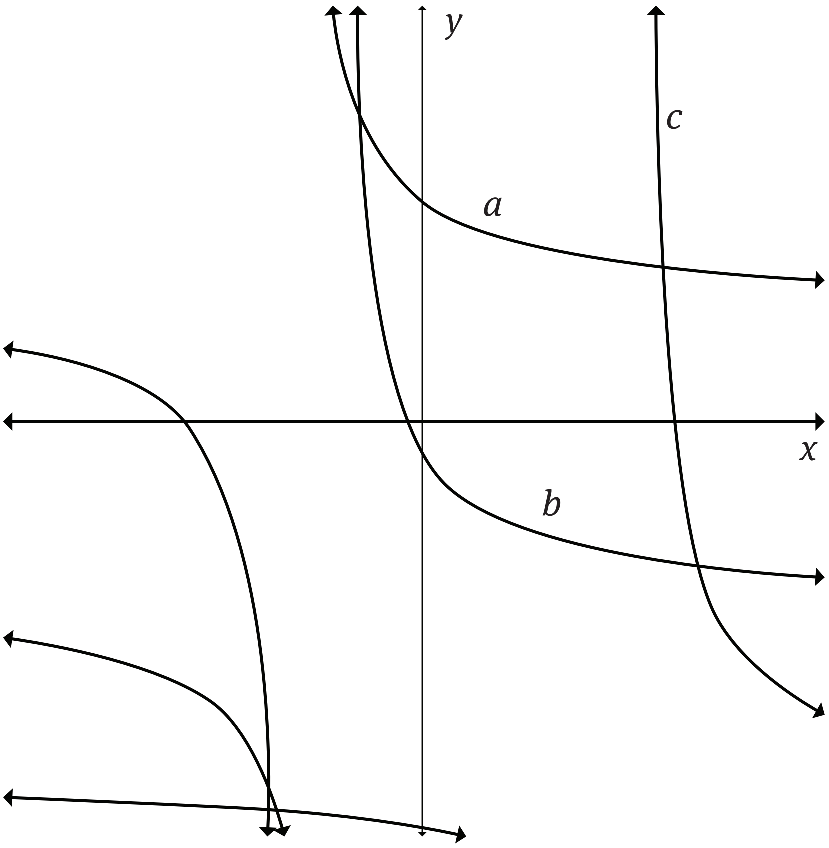
7.3 $y = -3/(x - 3) - 3$

7.4 $y = -5(x + 7)^{-1} + 12{,}5$

Question 8

Study the graphs of $f(x) = 5/(x - 5) - 6$, $g(x) = 5/(x + 3) + 2$ and $h(x) = 5/(x + 2) - 3$.

Match the graphs labelled a), b) and c) to any one of the functions f, g or h.



Question 9

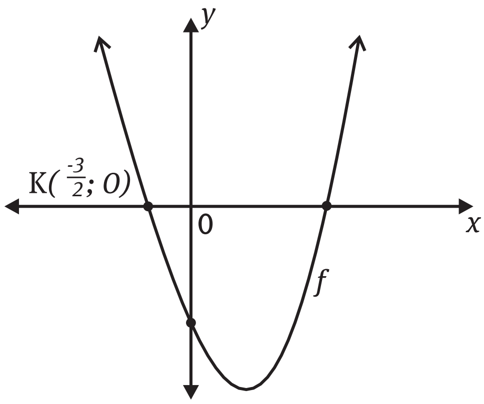
Draw sketch graphs of each group of functions on the same set of axes. Indicate the coordinates of the intercepts with the axes, as well as the equations of the horizontal asymptotes. Describe the translations that the basic graph has undergone after each change to the equation.

9.1	$y = -7^x$	$y = -7^x + 4$	$y = -7^{x+3} + 4$	$y = -7^{x-7} + 4$
9.2	$y = (\frac{1}{2})^x$	$y = (\frac{1}{2})^x - 3$	$y = (\frac{1}{2})^{x+2} - 3$	$y = (\frac{1}{2})^{x-3} - 3$
9.3	$f(x) = 3^{-x}$	$g(x) = 3^{-x} + 5$	$h(x) = 3^{-(x-1)} + 5$	$k(x) = 3^{(-x+1)} + 5$

Question 10

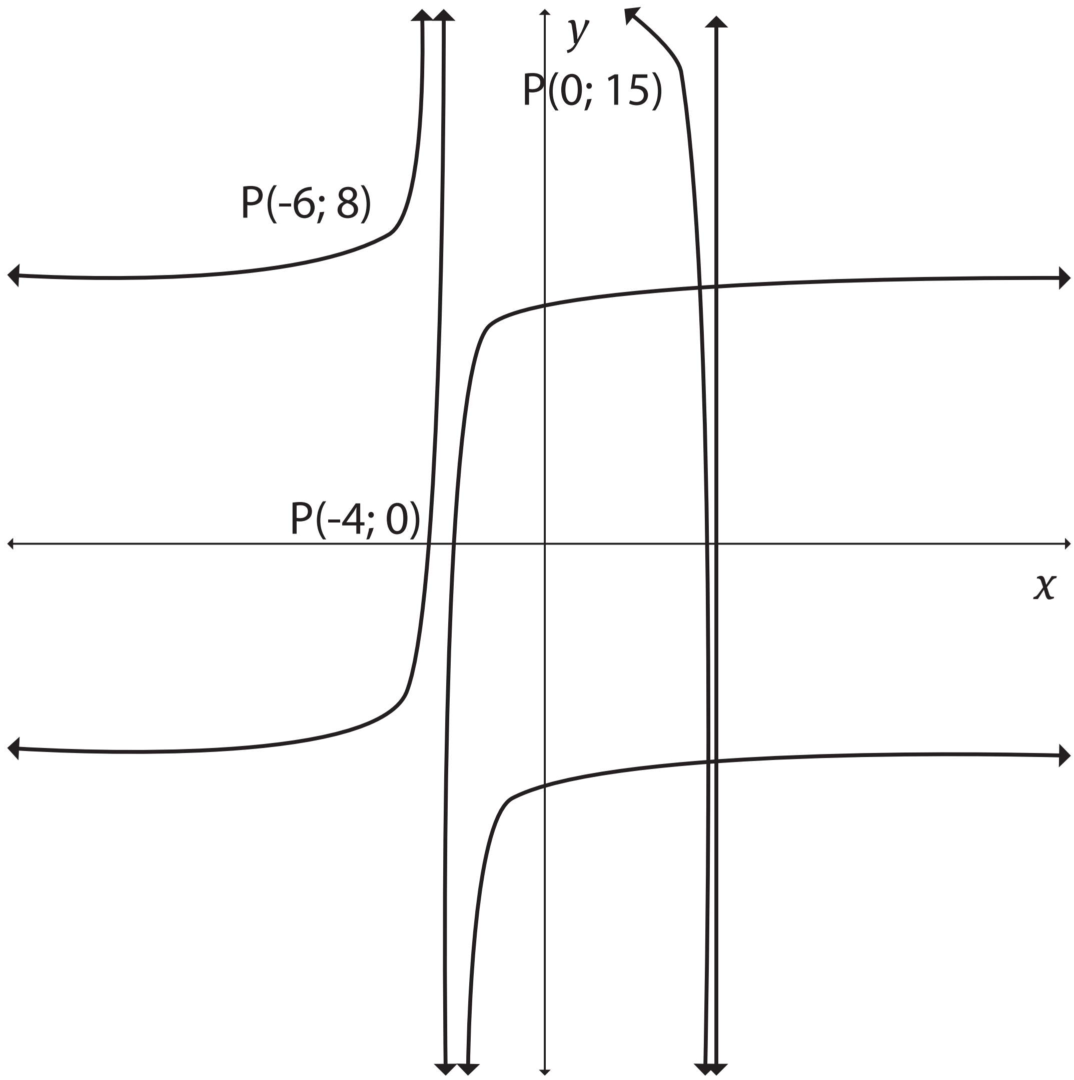
The graph of the function $f(x) = (x + p)^2 + q$ is sketched below.

K($-\frac{3}{2}$; 0) is an x-intercept. Determine the equation of the parabola.



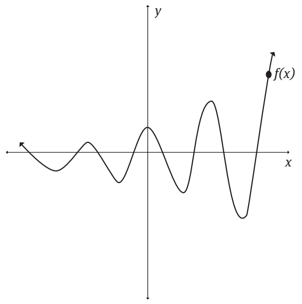
Question 11

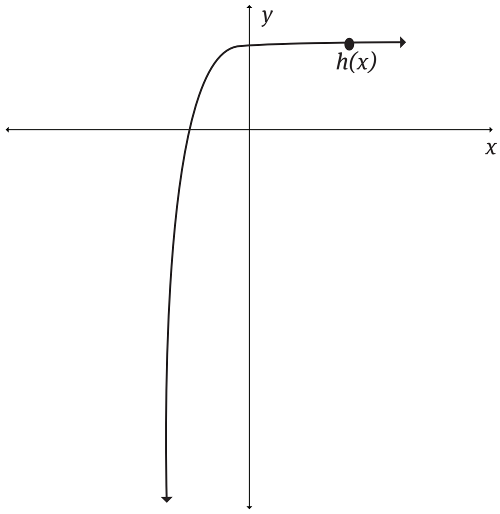
Graphs of the form $h(x) = a/(x + p) + q$ are sketched below. The graphs pass through the points P as indicated. Determine the equations of the graphs.

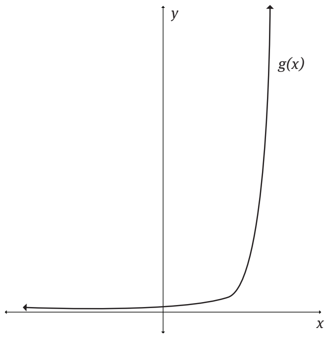


Question 12

Graphs of the form $f(x) = b^x$, $g(x) = a.5^{-x}$ and $h(x) = -a^x + q$ are sketched below. Determine the equation of each graph.

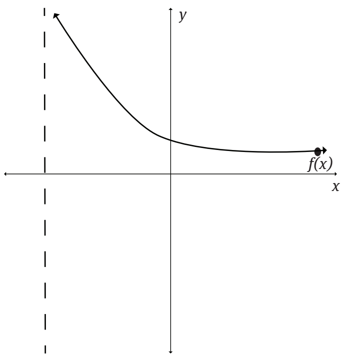


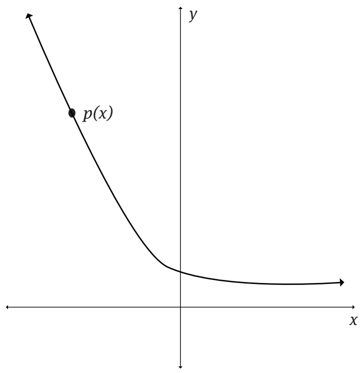




Question 13

Given the graphs of $f(x) = 5/x + 3$ and $p(x) = (\frac{1}{2})^x$.





13.1 Determine the length of KL.

13.2 Determine the length of KB, where B is the y-intercept of p.

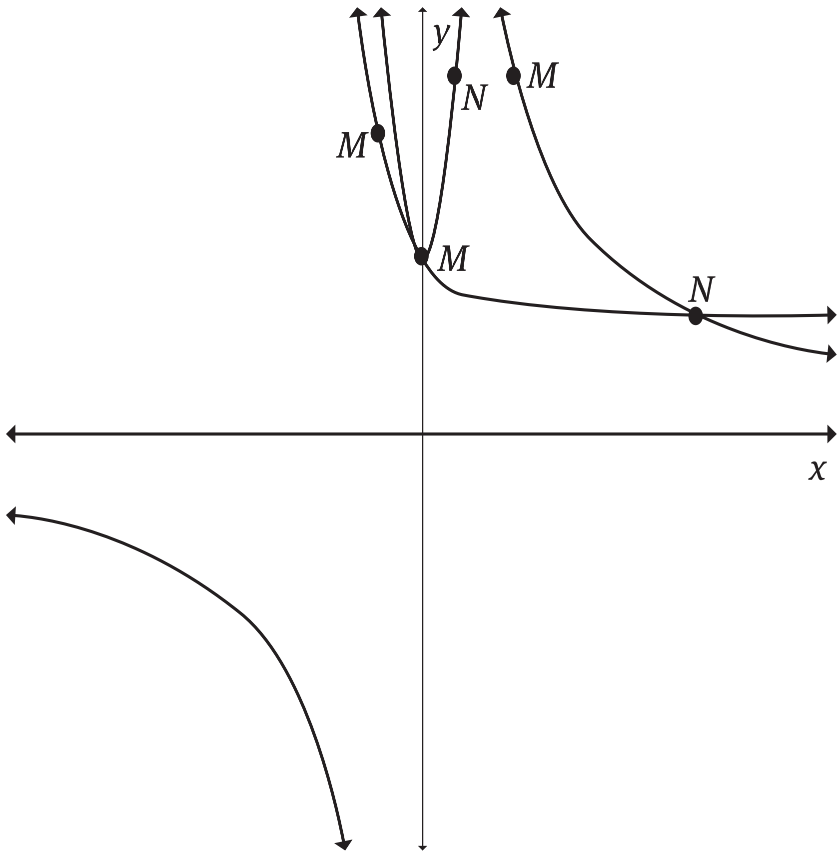
13.3 For which values of x, $x > 0$, is $f(x) = p(x)$?

13.4 For which values of x, $x > 0$, is $p(x) > f(x)$?

Question 14

Calculate the average gradient between M and N on the graphs of the functions sketched below:

14.1 $f(x) = 5x^2 + 3$ 14.2 $g(x) = 12{,}5/x$ 14.3 $h(x) = (1/3)^x + 2$



Question 15

A school needs to repaint its hall, and the Grade 11 class has volunteered to help with this. The table below gives the number of hours that it takes different numbers of learners to paint the hall:

Number of learners (x)	1	3	5	10	15
Hours taken (y)	15	5	3	1,5	1

15.1 The equation of the function that represents this relationship is of the form $y = a/x$. Determine the value of a.

15.2 Draw a graph of the function.

15.3 How will the graph be influenced, if the headmistress makes provision for 15 extra hours?

15.4 The domain of the function is restricted. Write down the restriction, and give a reason for the restriction.

Question 16

Draw neat sketch graphs of the groups of functions given below on the same system of axes for the domain given. Indicate the coordinates of the turning points (where applicable) as well as the intercepts with the axes. Remember to indicate the asymptotes where applicable.

16.1 $f(\theta) = 3\sin\theta$; $g(x) = -3\sin\theta$; $h(x) = -3\sin\theta - 3$; $0° \leq \theta \leq 360°$

16.2 $f(x) = 5\cos x$; $g(x) = -5\cos x$; $h(x) = -5\sin x + 2$; $-180° \leq \theta \leq 180°$

16.3 $f(x) = \tan x$; $g(x) = \tan x - 3$; $h(x) = \tan x + 5$; $0° \leq \theta \leq 360°$

Question 17

Copy and complete the table. The domain for all the graphs is $0° \leq \theta \leq 360°$.

	Highest y-value	**Lowest y-value**	**Amplitude**	**Range**	**Perio d**	**Ordered number pair at 45°**
$y = 3\sin x + 5$						
$y = \cos x - 5$						
$y = -\tan x - 3$						
$y = -4\cos x + 7$						
$y = \tan x + 5$						
$y = -5\sin x + 2$						

Question 18

Draw the graphs of the given functions for the indicated domains:

18.1 $y = 3\sin x,$ $x \in [-180°;\ 180°]$

18.2 $y = -3\cos x + 4,$ $x \in [0°;\ 180°]$

18.3 $y = 3\tan x - 2,$ $x \in [-360°;\ 360°]$

Question 19

Sketch the graphs of the functions given below, for the given domain, on separate axes.

Clearly indicate the coordinates of the intercepts with the axes, as well as those of the turning points.

Write down the period, amplitude and range of each graph.

19.1 $g(x) = \sin\left(\frac{1}{3}\right)x,$ $x \in [-360°;\ 360°]$

19.2 $f(x) = \cos 3x,$ $x \in [-180°;\ 180°]$

19.3 $k(x) = -\tan 5x,$ $x \in [0°;\ 360°]$

Question 20

Draw the graphs of the functions indicated on separate sets of axes. The domain for each graph is given.

Where applicable, indicate the:

- intercepts with the axes
- coordinates of the turning points
- position of the asymptote(s) with a dotted line.

20.1 $y = \sin(x + 30°),$ $x \in [-30°; 330°]$

20.2 $y = \cos(x - 45°),$ $x \in [-45°; 315°]$

20.3 $y = \tan(x + 60°),$ $x \in [-60°; 300°]$

20.4 $y = \sin(x + 20°),$ $x \in [-160°; 160°]$

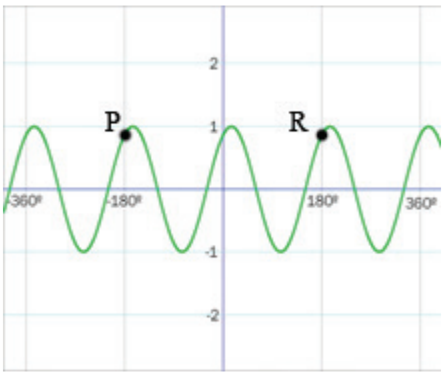
Question 21

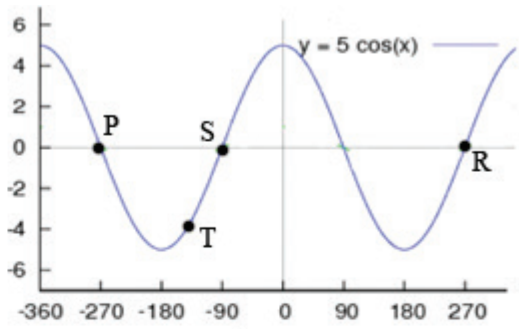
Match the function in column A to the correct description in column B.

	Column A		Column B
A	$y = 2\ \sin 3x$	**E**	The graph of $y = \sin x$ is shifted upwards by three units.
B	$y = \sin x + 3$	**F**	The graph has a period of 180° and is shifted 30° to the right.
C	$y = \sin 2(x - 30°)$	**G**	The amplitude is 2 and the period 120°.

Question 22

The graphs of $g(x) = \sin 2(x - 30°)$ and $p(x) = 5\cos x$ for $x \in [-180°;\ 90°]$ are sketched below.





22.1 Write down the range of p.

22.2 Write down the coordinates of R if the coordinates of P are $(-180°;\ 0{,}866)$ and $(-270°;\ 0)$ respectively.

22.3 Write down the period of $g(2x)$.

22.4 Determine the length of ST correct to four decimal places.

22.5 For which values of x is $g(x) = p(x)$?

22.6 For which values of x is $g(x) \leq p(x)$?

22.7 Write down the equation of h if g is shifted 45° to the right and 2,5 units downwards.

Chapter 6

Trigonometry

Overview

Chapter 6 Page 63 Trigonometry	**Unit 1 Page 64**	
	Trigonometric identities	• The fundamental trigonometric identities
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	Applying the trigonometric identities	• Simplifying trigonometric expressions • Proving identities
	Unit 3 Page 68	
	Reduction formulae	• Trigonometric ratios of $90° - \theta$ • Trigonometric identities of $90° + \theta$ • Applying the reduction formulae • Trigonometric ratios of $180° \pm \theta$ and $360° \pm \theta$
	Unit 4 Page 71	
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	Unit 5 Page 72	
	Solving trigonometric equations	• Using a graph • Using a calculator • Using special angles • General solutions • More complicated equations

Trigonometry deals with the relations of the sides and angles of triangles, and the relevant functions of any angles. In this chapter you will revise ratios and special angles. You will learn how to use and apply trigonometric identities. You will also learn about reduction formulae and how to work with negative angles and solve trigonometric equations.

Trigonometric identities

A trigonometric identity is an expression that is always true!

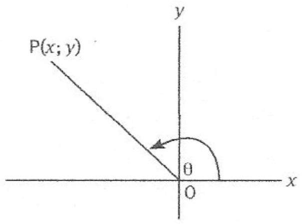
1.1 The fundamental trigonometric identities

- **Quotient identity:** $\tan\theta = \frac{\sin\theta}{\cos\vartheta}$

  (defined for all values of θ, except where $\cos\theta = 0$).

  Proof:

  Let $P(x;\ y)$ be any point on the terminal ray of θ in the standard position.



$\frac{\sin\theta}{\cos\theta} = (\frac{y}{r})/(\frac{x}{r})$ [read the ratios from the diagram]

$= \frac{y}{r} \times \frac{r}{x}$ [simplify the fraction]

$= \frac{y}{x} = \tan\theta$

- **Square identity:** $\sin^2\theta + \cos^2\theta = 1$ (also called the Pythagorean identity)

  Proof:

$\sin^2\theta + \cos^2\theta = (\frac{y}{r})^2 + (\frac{x}{r})^2$ [read the ratios from the diagram]

$= y^2/r^2 + x^2/r^2$ [square the terms]

$= (y^2 + x^2)/r^2$ [add the fractions]

$= r^2/r^2 = 1$ [apply the theorem of Pythagoras]

The square identity can also be expressed in the following ways:

- $\sin^2\theta = 1 - \cos^2\theta$
- $\cos^2\theta - 1 = -\sin^2\theta$
- $\cos^2\theta = 1 - \sin^2\theta$
- $\sin^2\theta - 1 = -\cos^2\theta$

Applying the trigonometric identities

2.1 Simplifying trigonometric expressions

- To simplify trigonometry we use the two fundamental identities (the quotient and square identities) and basic algebra.

Example 1

1 Simplify $\cos^2\theta.\tan\theta + \tan\theta.\sin^2\theta$

$$\cos^2\theta.\tan\theta + \tan\theta.\sin^2\theta = \tan\theta\,(\cos^2\theta + \sin^2\theta)$$
$$= \tan\theta\,(1)$$
$$= \tan\theta$$

2 Simplify $(2\tan b.\cos b.\sin b)/(\cos^2 b - 1)$

$$(2\tan b.\cos b.\sin b)/(\cos^2 b - 1) = 2\tan b.\cos b.\sin b/-\sin^2 b$$
$$= (2\tan b.\cos b)/(-\sin b)$$
$$= -(2(\sin b/\cos b).\cos b)/(\sin b)$$
$$= (-2\sin b)/(\sin b)$$
$$= -2$$

2.2 Proving identities

- There are many more identities than the two fundamental identities we have discussed so far.
- An identity can be proven by showing that the expression on the left-hand side (LHS) equals the expression on the right-hand side (RHS), and using the fundamental identities and basic algebra.
- There are also other ways to prove identities. They are:

Simplify one side to equal the other side

Prove the identity: $\tan a + \cos a/(1 + \sin a) = 1/\cos a$

LHS $= \tan a + \cos a/(1 + \sin a)$
$= \sin a/\cos a + \cos a/(1 + \sin a)$
$= (\sin a(1 + \sin a) + \cos^2 a)/\cos a(1 + \sin a)$
$= (\sin a + \sin^2 a + \cos^2 a)/\cos a(1 + \sin a)$
$= (\sin a + 1)/\cos a(1 + \sin a)$
$= 1/\cos a$
= RHS

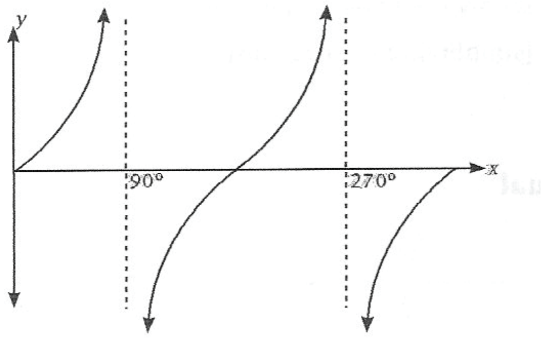
Simplify both sides until they are equal

Prove that $\sin^2 y - (\cos^2 y)/(\sin y \cos y + \cos 2y) = \tan y - 1$

LHS $= (\sin^2 y - \cos^2 y)/(\sin y \cos y + \cos 2y)$
$= ((\sin y - \cos y)(\sin y + \cos y))/(\cos y(\sin y + \cos y))$
$= (\sin y - \cos y)/\cos y$
RHS $= \tan y - 1$
$= \sin y/\cos y - 1$
$= (\sin y - \cos y)/\cos y$
∴ LHS = RHS

- Trigonometric ratios are not always true for all the values of the variables. The exceptions are when the identity contains a tangent function, or when there is a fraction with a trigonometric function in the denominator.

Identities containing tangent functions



- The function $y = \tan x$ is undefined in the interval $[0°;\ 360°]$ for $x = 90°$ or $270°$.

Prove that: $\tan^2 k.\cos^2 k = \sin^2 k$
LHS = $\tan^2 k.\cos^2 k = (\sin^2 k/\cos^2 k).\cos^2 k = \sin^2 k$ =RHS.
The identity is undefined for $k = 90°$ or $270°$.

Identities with trigonometric functions in the denominator

When identities have trigonometric functions in the denominator, you have to remember that the denominator can never be equal to zero. This means that for all values of the variable for which the trigonometric function will equal zero, the identity is invalid.

Prove that $(1 - \sin^3 p)/(1 - \sin p) = 1 + \sin p + \sin^2 p$

LHS $= (1 - \sin^3 p)/(1 - \sin p)$
$= ((1 - \sin p)(1 + \sin p + \sin^2 p))/(1 - \sin p)$
$= 1 + \sin p + \sin^2 p$
= RHS.

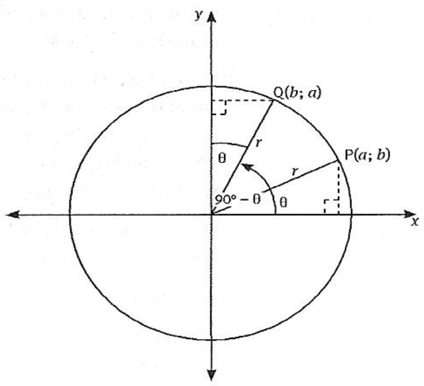
The identity is undefined when $1 - \sin p = 0$

∴It is undefined when $\sin p = 1$.

Reduction formulae

Here we derive formulae for the trigonometric ratios of $90° - \theta$ and $90° + \theta$. We let θ be an acute angle. What we want to prove is that there is a relationship between the trigonometric ratios of θ and that of $90° - \theta$ or $90° + \theta$ or $180° - \theta$ or $180° + \theta$ or $360° - \theta$ or $360° + \theta$. You need to study the formal proofs and the formulae.

3.1 Trigonometric ratios of 90° – θ



If P(a; b) is a point on the terminal arm of θ in standard position, and Q is a point on the terminal arm of $90° - \theta$, then Q is a reflection of P about the line $y = x$. Therefore, the coordinates of Q will be (b; a) and the length of oP = oQ = r.

Thus: $\sin(90° - \theta) = a/r = \cos\theta$

$\cos(90° - \theta) = b/r = \sin\theta$

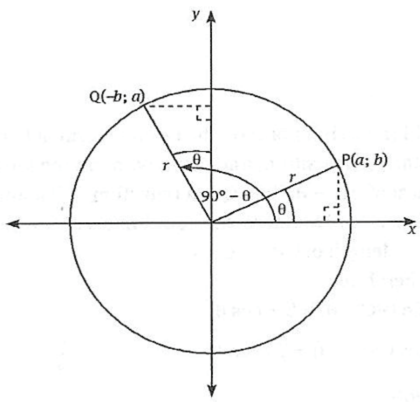
Remember: $y = \sin x$ and $y = \cos x$ are co-functions.

Angles θ and $90° - \theta$ add up to $90°$ and are complementary angles.

The function values of $90° - \theta$ equal the co-function values of θ.

3.2 Trigonometric ratios of 90° + θ

If P(a; b) is a point on the terminal arm of θ in standard position and Q is a point on the terminal arm of $90° + \theta$, then Q is a rotation of P, $90°$ anticlockwise through the origin.
Therefore, the coordinates of Q will be ($-b$; a) and the length of oP = oQ = r.



Thus: $\sin(90° + \theta) = a/r = \cos\theta$

$\cos(90° + \theta) = b/r = -\sin\theta$

3.3 Applying the reduction formulae

Example 2

1 Express sin 54°, cos 126° and cos 54° as trigonometric ratios of 36°.

$\sin 54° = \sin(90° - 36°) = \cos 36°$

$\cos 126° = \cos(90° + 36°) = -\sin 36°$

$\cos 54° = \cos(90° - 36°) = \sin 36°$

2 Simplify: $\sin(90° - \theta).\cos(90° + \theta).\tan\theta + \tan 225°$

$\sin(90° - \theta).\cos(90° + \theta).\tan\theta + \tan 225° = \cos\theta.(-\sin\theta).(\sin\theta/\cos\theta) + 1$

$= -(\sin^2\theta.\cos\theta/\cos\theta) + 1$

$= -\sin^2\theta + 1$

$= -(\sin^2\theta - 1)$

$= -\cos^2\theta$

3 Evaluate without using a calculator: $\sin 68°/\cos 22°$

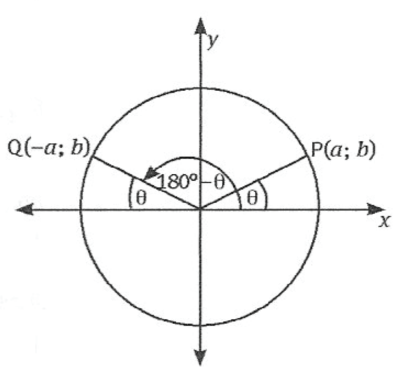
$\sin 68°/\cos 22° = \sin(90° - 22°)/\cos 22°$

$= \cos 22°/\cos 22°$

$= 1$

3.4 Trigonometric ratios of 180° ± θ and 360° ± θ

- Trigonometric ratios of $180° - \theta$



If P(a; b) is a point on the terminal arm of θ in the standard position and Q is a point on the terminal arm of $180° - \theta$, then Q is a reflection of P about the y-axis (the line $x = 0$). The coordinates of Q will therefore be ($-a$; b). The length of oP = oQ = r.

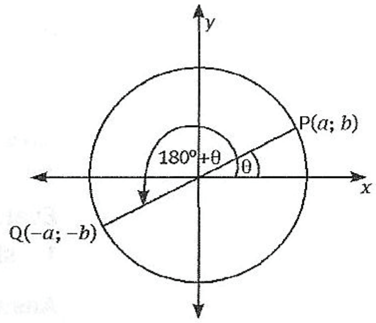
Thus: $\sin(180° - \theta) = b/r = \sin\theta$

$\cos(180° - \theta) = -a/r = -\cos\theta$

$\tan(180° - \theta) = b/-a = -\tan$

- Trigonometric ratios of $180° + \theta$

  If P(a; b) is a point on the terminal arm of θ in the standard position and Q is a point on the terminal arm of $180° + \theta$, then Q is a rotation of P, 180° through the origin. The coordinates of Q will therefore be (a; $-b$). The length of oP = oQ = r.

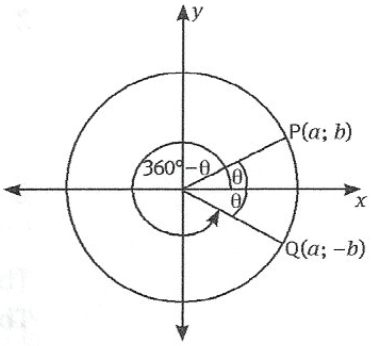


  Thus: $\sin(180° + \theta) = -b/r = -\sin\theta$

  $\cos(180° + \theta) = -a/r = -\cos\theta$

  $\tan(180° + \theta) = -b/a = \tan\theta$

- Trigonometric ratios of $360° - \theta$



  If P(a; b) is a point on the terminal arm of θ in the standard position and Q is a point on the terminal arm of $360° - \theta$, then Q is a rotation of P, 180° through the origin. The coordinates of Q will therefore be ($-a$; $-b$). The length of oP = oQ = r.

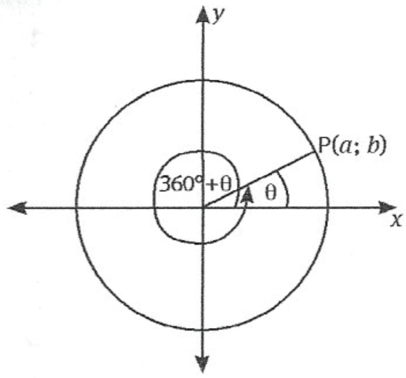
  Thus: $\sin(360° - \theta) = -b/r = -\sin\theta$

  $\cos(360° - \theta) = a/r = \cos\theta$

  $\tan(360° - \theta) = -b/a = -\tan\theta$

- Trigonometric ratios of $360° + \theta$

  If a point P(a; b) on the terminal arm of θ in the standard position is rotated 360° through the origin, then the resulting point on the terminal arm of $360° + \theta$ will coincide with P(a; b) on the terminal of θ in the standard position. The length of oP = r.



  Thus: $\sin(360° + \theta) = b/r = \sin\theta$

  $\cos(360° + \theta) = a/r = \cos\theta$

  $\tan(360° + \theta) = b/a = \tan\theta$

Here is a summary of the reduction formulae:

Reduction formulae			
$180° - \theta$	**$180° + \theta$**	**$360° - \theta$**	**$360° + \theta$**
$\sin(180° - \theta) = \sin\theta$	$\sin(180° + \theta) = -\sin\theta$	$\sin(360° - \theta) = -\sin\theta$	$\sin(360° + \theta) = \sin\theta$
$\cos(180° - \theta) = -\cos\theta$	$\cos(180° + \theta) = -\cos\theta$	$\cos(360° - \theta) = \cos\theta$	$\cos(360° + \theta) = \cos\theta$
$\tan(180° - \theta) = -\tan\theta$	$\tan(180° + \theta) = \tan\theta$	$\tan(360° - \theta) = -\tan\theta$	$\tan(360° + \theta) = \tan\theta$

Negative angles

In trigonometry we measure angles in the Cartesian plane from the positive x-axis.

Positive angles are measured in an anticlockwise direction from this axis, and negative angles in a clockwise direction.

4.1 Applying negative angles

Example 3

1 Express the following as trigonometric ratios of θ:

1.1 $\cos(-\theta) = \cos\theta$

1.2 $\tan(\theta - 180°) = \tan[-(180° - \theta)]$

$= -\tan(180° - \theta)$

$= -(-\tan\theta)$

$= \tan\theta$

1.3 $\cos(180° - \theta) = -\cos\theta$

2 Evaluate without using a calculator

2.1 $\sin(-90°) = -\sin 90° = -1$

2.2 $\cos(-120°) = \cos(120°)$

$= \cos(90° + 30°)$

$= -\sin 30°$

$= -\frac{1}{2}$

2.3 $\sin(-x)\cos(90° - x) - \cos^2 x = -\sin x.\sin x - \cos^2 x$

$= -(\sin^2 x + cos^2 x)$

$= -1$

Solving trigonometric equations

We can solve trigonometric equations by using a graph, a calculator or special angles.

5.1 Using a graph

- When you are given a graph and asked to solve the trigonometric equation, the solution is simple – read the value off the graph!
- It is important to remember that the equation might have more than one solution on the given interval, so watch out for that.

5.2 Using a calculator

- If you are given a trigonometric equation, e.g. $\tan x = -2{,}5$, it is important to first identify in which quadrant(s) the equation will hold.
- Once you have the quadrant(s), calculate the value of the variable and apply that to the trigonometric ratio(s) that hold in that (those) quadrant(s).

5.3 Using special angles

- If you are given a trigonometric equation with a reference angle, find the sign of the RHS of the equation. From that, determine in which quadrants the equation will be true (that is, in which quadrants there are solutions).
- Now, use the reference angle and the quadrants in which the solutions are, and calculate the value(s) of the variable.

5.4 General solutions

- The sine ratio
  The general solution to $\sin \alpha = 0{,}5$ is
  $\theta = 90° + k.360°$ or $\theta = 270° + k.360°,\ k \in Z.$
- The cosine ratio
  The general solution to $\cos \alpha = -\sqrt{3/2}$ is
  $\theta = 120° + k.360°$ or $\theta = 240° + k.360°,\ k \in Z.$
- The tan ratio
  The general solution to $\tan \alpha = 1$ is
  $\alpha = 45° + k.180°,\ k \in Z.$

5.5 More complicated equations

- Equations of the form $\sin\alpha = \pm\sin\beta$, $\cos\alpha = \pm\cos\beta$ and $\tan\alpha = \pm\tan\beta$.
  The following relationships between α and β are true, and can be used to solve these equations:
  - o $\tan\alpha = -\tan\alpha$
  - o $\sin\alpha = \sin\beta$
  - o $\sin\alpha = -\sin\beta$
  - o $\cos\alpha = \cos\beta$
  - o $\cos\alpha = -\cos\beta$
- Equations of the form $a\sin x = b\cos x$.
  - o Remember, this equation can be changed into a simple equation by dividing with $a\cos x$ on both sides. The equation becomes $\tan x = b/a$.
- Equations of the form $\sin A = \pm\cos B$.
  - o Use the co-function reduction formula to rewrite the equation in the form $\sin A = \sin(90° \pm B)$ or $\cos(90° \pm A) = \pm\cos B$.
  - o Now it is easy to solve the equation by using the same method we used for equations of the form $\sin a = \sin\beta$ and $\cos a = \cos\beta$.
- Quadratic equations are solved by using algebraic skills to factorise the quadratic equation into two first degree equations that you can solve easily!

Example 4

Solve for x, $x \in [0°; 360°]$ if $\tan x = -5{,}32$

The value of $\tan x = -5{,}32$, which implies that $\tan x$ is negative. Therefore the solutions are in the second and fourth quadrants. Recall that the angles in the second and third quadrants are of the form $180° - \theta$ or $360° + \theta$. To find the acute angle, θ, we calculate $\tan^{-1} - 5{,}32$, which equals $-79{,}35°$.

$\tan x = -5{,}32$

$\therefore\ x = -79{,}35°$

- Equations containing $\tan x$ together with $\sin x$ and $\cos x$.
  - o Express $\tan x$ in terms of $\sin x$ and $\cos x$ by using the quotient identity $\tan x = \sin x/\cos x$.

Example 5

Determine the general solution of $2\cos x + 3\tan x + 1 = -6\sin x$.

$2\cos x + 3\tan x + 1 = -6\sin x$

$2\cos x + 3\,(\sin x/\cos x) + 1 = -6\sin x$

$\therefore\ 2\cos^2 x + 3\sin x + \cos x = -6\sin x\cos x$

$\therefore\ 2\cos^2 x + \cos x + 6\sin x\cos x + 3\sin x = 0$

$\therefore\ \cos x(2\cos x + 1) + 3\sin x(2\cos x + 1) = 0$

$\therefore\ (2\cos x + 1)(\cos x + 3\sin x) = 0$

$\therefore \cos x = 1/2$ or $\cos x = 3\sin x$

$\cos x/3\cos x = 3\sin x/3\cos x$

$\frac{1}{3} = \tan x$

$\therefore\ x = 60° + 360°n$ or $360° - 60° + 360°n$

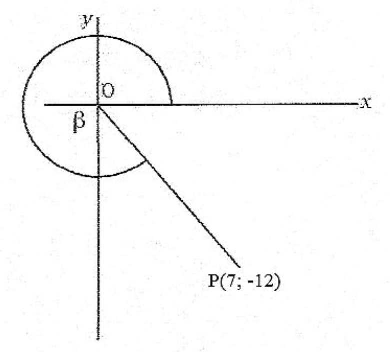
$\therefore\ x = 60° + 360°n$ or $300° + 360°n$

$\therefore\ x = 18{,}4° + 180°n$

Questions

Question 1

The point P(7; -12) lies on the terminal ray of β in standard position.



1.1 Determine:

1.1.1 the length of OP

1.1.2 $\sin\beta$

1.1.3 $\cos\beta$

1.1.4 $\sin\beta/\cos\beta$

1.1.5 $\tan\beta$

1.2 Use your answers in a 1.1.1 to 1.1.5 above to make a conjecture about the values of $\tan\beta$, $\cos\beta$ and $\sin\beta$.

1.3 Determine $\sin^2\beta + \cos^2\beta$.

Question 2

Use fundamental identities to simplify:

2.1 $\sqrt{(1 - \cos x)(1 + \cos x)}$

2.2 $\sin x(\sin x + \tan x . \cos x)$

2.3 $(\cos x + \sin x)^2 + (\cos x - \sin x)^2$

2.4 $\cos^2 y + (\sin^2 y . \tan y / \cos y)$

Question 3

Find the values of x, $x \in [0°; 360°]$, for which the identity is undefined.

Prove the identities.

3.1 $\cos t = 1/\cos t - \sin^2 t/\cos t$

3.2 $\sin^2 b - 2 = -\cos^2 b(\tan^2 b + 2)$

3.3 $\tan^2 ß = \sin^2 ß(1 + \tan^2 ß)$

3.4 $\tan x - \sin x . \cos x = \sin^2 x . \tan x$

3.5 $\sin x - 1/\sin x = -\cos^2 x/\sin x$

3.6 $(\sin^3 x - \cos^3 x)/(1 + \sin x \cos x) = \sin x - \cos x$

Question 4

4.1 Express each of the following as a function of $42°$:

4.1.1 $\cos 222°$

4.1.2 $\sin 48°$

4.2 Express each of the following as a function of $75°$:

4.2.1 $\sin 435°$

4.2.2 $\cos 285°$

Question 5

If $\sin 88° = t$, express the following in terms of t. Use a sketch.

5.1 $\cos 2°$

5.2 $\tan 178°$

5.3 $\cos 92°$

Question 6

Express each of the following as a trigonometric ratio of θ:

6.1 $\tan(360° + \theta)$

6.2 $\cos(90° - \theta)$

6.3 $\sin(360° - \theta)$

6.4 $\tan(360° - \theta)$

Question 7

Evaluate without using a calculator:

7.1 $\sin 150°$

7.2 $\cos 300°$

7.3 $\tan 390°$

7.4 $\cos 210° + \sin 120°$

7.5 $\sin(90° - x).\cos(180° + x).\tan(180° + x)/\cos(90° - x).\cos(360° - x).\tan x$

7.6 $\sin(-\theta)$

7.7 $\cos(-360° - \theta)$

Question 8

Prove: $\cos x - \cos x.\tan(-x) = \cos(-x) - \sin(x - 180°)$.

Question 9

Determine the general solutions of the following equations.

Give your answers correct to one decimal digit.

9.1 $\sin x = -3{,}125$

9.2 $\cos x - 0{,}986 = 0$

9.3 $3\tan x = 2{,}12$

9.4 $\tan x = \sin 35°$

9.5 $2\cos x + 1 = 0$

9.6 $\tan 4x = \tan 90°$

9.7 $2\tan x.\cos^2 x = \cos x$

9.8 $\sin(5x + 30°) = \sin x$

9.9 $\tan(3x - 20°) = \tan(x + 40°)$

9.10 $3\sin x = -5\cos x$

9.11 $\frac{5}{9}\cos x = -\sin x$

9.12 $\cos(x - 20°) = -\sin(x + 30°)$

9.13 $\cos(3x + 20°) = -\sin(x - 60°)$

9.14 $\sin^2 A - 2\sin A.\cos A = 2\cos^2 A$

9.15 $3\tan x = 2\sin x$

9.16 $-3 - \tan x = -\cos x/\sin x$

Overview

Chapter 7 Page 77 Measurement	Unit 1 Page 78 Combined objects

Surface area and volume can be used in many different situations to help us save time, save money and save the environment.

Geysers use a lot of energy to heat water for household use. Using calculations involving the volume of water used by a household, the optimal geyser size can be installed that will ensure that no unnecessary energy is wasted to heat large amounts of water.

To paint the inside walls of a house we calculate the area in m^2 and purchase exactly enough paint to cover the walls.

If we need to construct a wooden crate, we must decide beforehand what the volume needs to be. We are then able to work out the optimal amount of wood needed so that we can purchase the correct sizes and waste as little as possible.

Combined objects

Let's revise some of the definitions and formulae used to calculate the area and volume of three-dimensional (3D) objects:

- A right prism is a polyhedron with two identical and parallel faces, or bases. The other faces are rectangles and are perpendicular to the bases.
- A right cylinder is a 3D object with two circular bases which are parallel to each other. There is a curved surface between the two bases.
- The surface area of a right prism and a right cylinder is given by
  $\text{S} = 2(\text{area of base}) + \text{perimeter of base} \times \text{height of prism (cylinder)}$
- The volume of a right prism and a right cylinder is given by
  $\text{V} = \text{area of base} \times \text{height of prism (cylinder)}$

Prisms

- When one dimension of a prism is multiplied by a factor, k, the volume of the prism changes by the same factor.
- When two dimensions of a prism are multiplied by a factor, k, the volume of the prism changes by a factor of k^2.
- When three dimensions of a prism are multiplied by a factor, k, the surface area changes by a factor of k^2, and the volume changes by a factor of k^3.

Cylinders

- When the radius and the height of a cylinder are multiplied by a factor, k, the surface area of a cylinder changes by a factor of k^2.
- When we multiply the height of a cylinder by a factor, k, the volume of the cylinder changes by a factor of k.
- When the radius or diameter of a cylinder is multiplied by a factor, k, the volume of the cylinder changes by a factor of k^2.
- When both the radius (or diameter) and the height of a cylinder are multiplied by a factor, k, the volume of the cylinder changes by a factor of k^3.

Spheres (r = radius, d = diameter (= 2r))

- Circumference of a sphere: $C = 2\pi r = \pi d$
- Surface area of a sphere: $S = 4\pi r^2 = \pi d^2$
- Volume of a sphere: $V = \frac{4}{3}\pi r^3 = \pi d^3/6$

Cones (r = radius of circular base, h = height of cone, s = slant height of cone)

- Slant height of cone: $s = \sqrt{h^2 + r^2}$
- Circumference of base: $C = 2\pi r = \pi d$
- Surface area of a cone: $S = \pi r^2 + \pi rs$
- Volume of a cone: $V = 1/3\, \pi r^2 h$

Pyramids (h = height of pyramid, s = slant height of pyramid)

- Surface area of a pyramid: $S = \text{area of base} + 1/2\ (\text{perimeter of base})s$
- Volume of a pyramid: $V = 1/3\ (\text{area of base})h$

Questions

Question 1

1.1 Calculate correct to three decimal places, the surface area and volume of a sphere with radius 12 mm.

1.2 Calculate the radius and volume of a sphere with surface area of 512,157 m^2.

Question 2

The radius of the base of a right circular cone is 13,5 cm and the height of the cone is 51,2 cm. Determine, correct to three decimal places:

2.1 the slant height of the cone

2.2 the total surface area of the cone

2.3 the volume of the cone.

Question 3

A tower is made from the following parts:

A right pyramid sits on top of a squared right prism.

The base of the tower is 2,5 m by 2,5 m.

The total height of the building is 50 m.

If the pyramid is 23 m high, calculate the external surface area of the tower, correct to the nearest square meter.

Question 4

A flowerpot is constructed with a hemisphere at the bottom, with a right cylinder on top of it. The radius of the hemisphere is 13 cm and the height of the cylinder is 31 cm. Calculate the surface area and volume of the flowerpot correct to two decimal places if necessary.

Do you think this is a practical design? Why do you think so?

Chapter 8

Euclidean geometry

Overview

Chapter 8 Page 81 Euclidean Geometry	Grade 10 Revision Page 82	
	Unit 1 Page 84 Circles	• Circle theorems and corollaries
	Unit 2 Page 87 Cyclic quadrilaterals	• Cyclic Quadrilateral Theorems and Corollaries • Proving a quadrilateral is cyclic
	Unit 3 Page 89 Tangents to a circle	

In this chapter you will revise everything you have learnt in Grade 10 about intersecting lines, parallel lines, triangles and quadrilaterals.

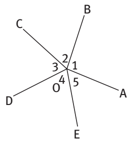
Euclid, a Greek mathematician who lived more than 2 000 years ago, is also known as the *Father of Geometry*. His book *Elements* was used as the main textbook for teaching Mathematics until the early twentieth century!

Grade 10 revision

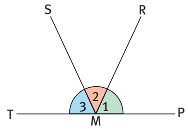
Revision of Grade 10

Intersecting lines

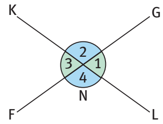
The sum of the angles around a point is 360°.



Adjacent angles at a point on a line segment are supplementary.

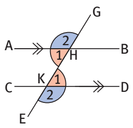


When two line segments intersect, the vertically opposite angles are equal.

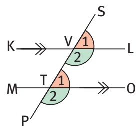


Parallel lines

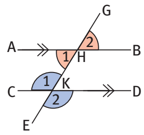
When a transversal cuts parallel lines, the alternate interior angles and the alternate exterior angles are equal.



When a transversal cuts two parallel lines, the corresponding angles are equal.

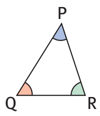


When a transversal cuts parallel lines, the co-interior and the co-exterior angles are supplementary.

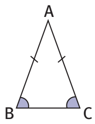


Triangles

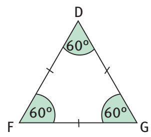
The interior angles of a triangle are supplementary.



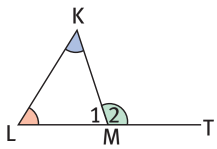
The base angles of an isosceles triangle are equal.



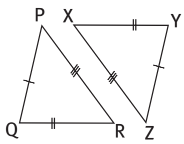
All interiors angles of an equilateral triangle are 60°.



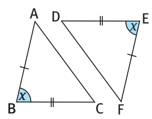
An exterior angle of a triangle is equal to the sum of the opposite interior angles.

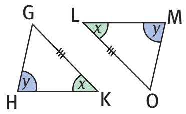
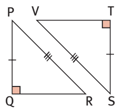
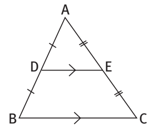
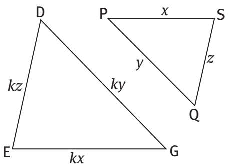
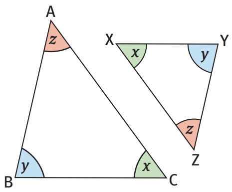
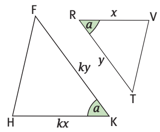
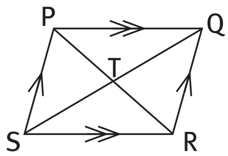
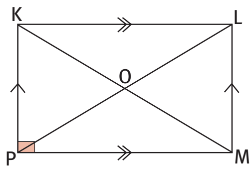
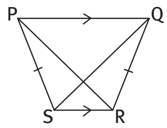
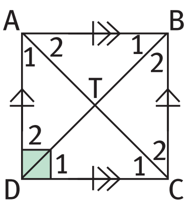
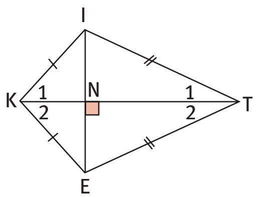
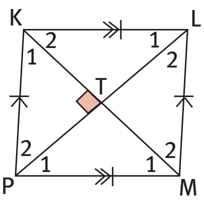


When the sides of one triangle are equal to the corresponding sides of another triangle, the two triangles are congruent. [SSS]



Two triangles are congruent when two sides and the inscribed angle of one triangle are equal to two sides and the inscribed angle of the other triangle. [SAS]



Triangles		
When two angles and a side of one triangle are equal to two angles and the corresponding side of another triangle, the two triangles are congruent. [AAS] 	Two triangles are congruent when the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and one side of another right-angled triangle. [90°HS] 	The line segment joining the midpoints of two sides of a triangle is parallel to the third side and half the length of the third side. [DE‖DB] 
When the corresponding sides of two triangles are in the same ratio, the two triangles are similar. [SSS] 	When the corresponding angles of two triangles are equal, the two triangles are similar. [AAA] 	When two pairs of sides of two triangles are in the same ratio and the angles between those sides are equal, the two triangles are similar. [SAS] 
Quadrilaterals		
A parallelogram is a quadrilateral with opposite sides parallel. [PQ‖SR, PS‖QR] 	A rectangle is a parallelogram with a 90° interior angle. [∠KPM = 90°] 	An isosceles trapezium is a quadrilateral with one pair opposite sides parallel and the other pair equal. [PQ‖SR, PS = QR] 
A square is a rhombus with a 90° interior angle. [AB = BC = CD = DA] 	A kite is a quadrilateral with two pairs of adjacent sides equal. [KI = KE, ET = TI] 	A rhombus is a parallelogram with all sides equal. [KL = LM = MP = PK] 

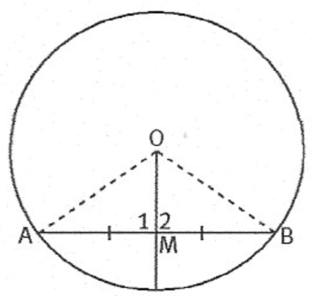
1.1 Circle Theorems and Corollaries

You need to know these theorems and corollaries, and how to apply them, in order to answer questions relating to circles. Don't worry, once you get the hang of it you will love solving these problems!

Remember to add abbreviated reasons for all theorems!

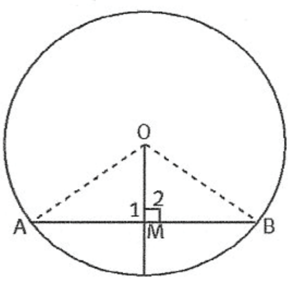
Theorem 1: Segment through centre and midpoint

- The line segment (radius) joining the centre of a circle and the midpoint of a chord is perpendicular to the chord.



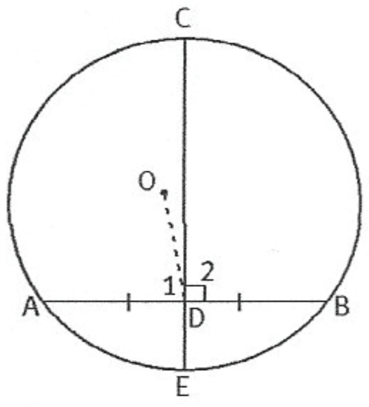
Theorem 2: Perpendicular from centre to chord

- A line segment (radius) from the centre of a circle perpendicular to a chord will bisect the chord.



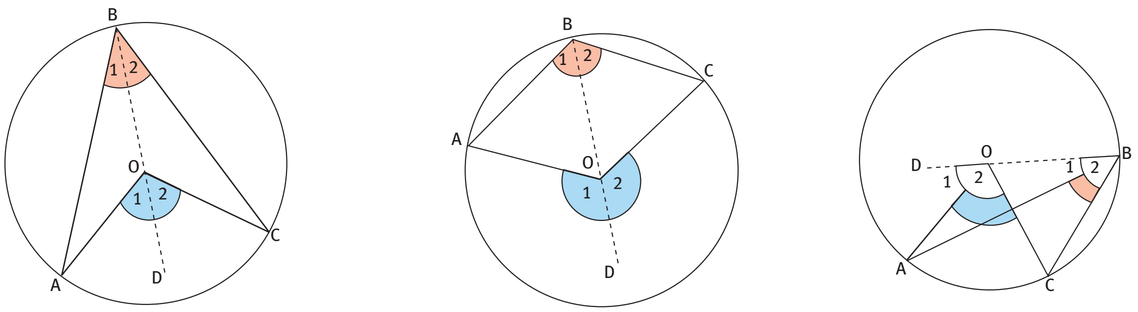
Theorem 3: Perpendicular bisector of chord

- The perpendicular bisector of a chord of a circle passes through the centre of the circle.



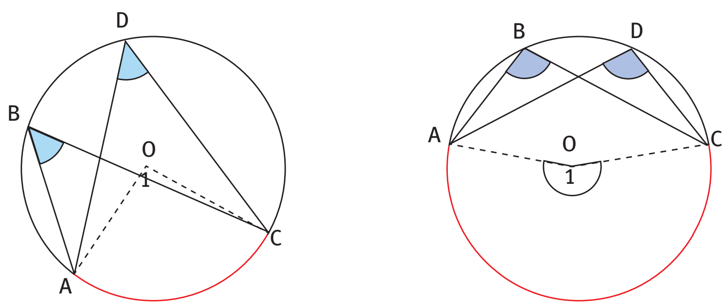
Theorem 4: Angle at the centre is equal to twice angle on circle

- An angle at the centre of a circle is twice the angle on the circle subtended by the same chord or arc.



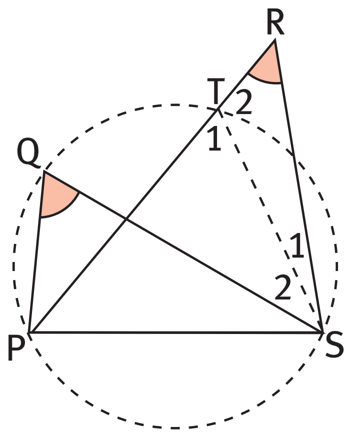
Theorem 5: Angles subtended by same arc (chord)

- Angles on a circle that are subtended on the same side by the same arc (chord), are equal.



Theorem 6: Line segment subtends equal angles

- (converse of Theorem 5)
- When a line segment joining two points subtends equal angles at two other points on the same side of the line segment, these four points are cyclic.



Corollaries

1. Equal chords (arcs) of a circle subtend equal angles at the centre of the circle.
2. Equal chords (arcs) of a circle subtend equal angles on the circle in the corresponding segments.
3. Chords (arcs) of a circle are equal when they subtend equal angles at points on the circle or at the centre.
4. A diameter of a circle subtends right angles on the circle.
5. Equal chords (arcs) in different circles with equal radii (diameter) subtend equal angles on the circles.
6. When the line segment joining two points subtend equal angles on the same side of it, those four points are cyclic (they lie on a circle).

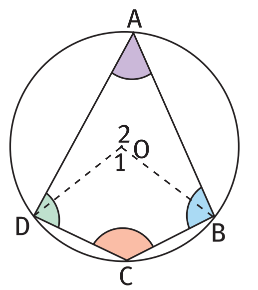
2.1 Cyclic Quadrilateral Theorems and Corollaries

What does cyclic mean? It means that a circle can be drawn through all vertices.

Remember to add abbreviated reasons for all theorems!

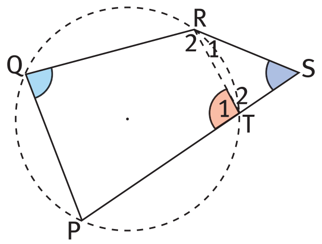
Theorem 1: Opposite interior angles of a cyclic quadrilateral

- The opposite interior angles of a cyclic quadrilateral are supplementary.



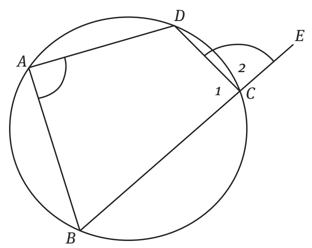
Converse of Theorem 1: Opposite interior angles supplementary

- When the opposite interior angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.



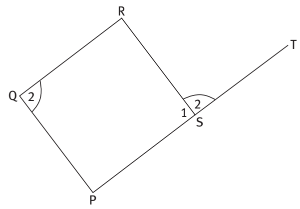
Theorem 2: Exterior angle of a cyclic quadrilateral

- An exterior angle of a cyclic quadrilateral is equal to the opposite interior angle of the cyclic quadrilateral.



Converse of Theorem 2: Exterior angle is equal to opposite interior angle

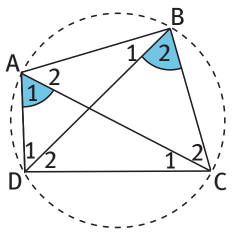
- When an exterior angle of a quadrilateral is equal to the opposite interior angle of the quadrilateral, then the quadrilateral is cyclic.



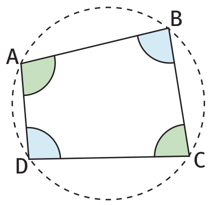
2.2 Proving a quadrilateral is cyclic

This can be done in any one of three ways:

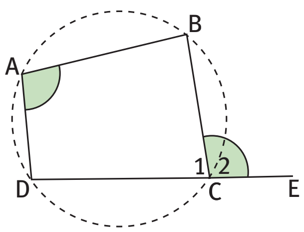
1. When any side of a quadrilateral subtends equal angles at the other vertices, the quadrilateral is cyclic.



2. When any two opposite interior angles of a quadrilateral are supplementary, the quadrilateral is cyclic.



3. When an exterior angle of a quadrilateral is equal to the opposite interior angle, the quadrilateral is cyclic.

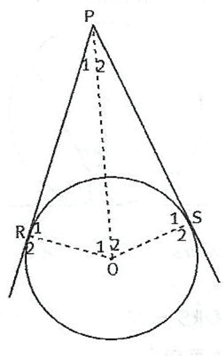


Tangents to a circle

- A tangent to a circle is a line that touches the circle at one point only.
- A tangent to a circle is perpendicular to the radius at the point of contact.

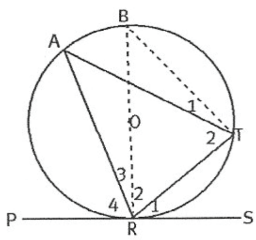
Theorem 1: Tangents from the same point

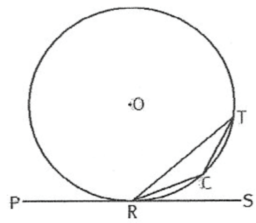
- Two tangents drawn to a circle from the same point outside the circle are equal in length.



Theorem 2: Angle between tangent and chord

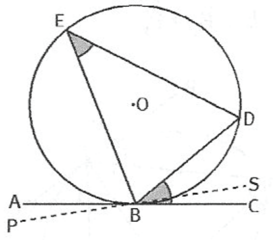
- The angle between a tangent to a circle and a chord drawn from the point of contact is equal to an angle on the circle subtended by the chord in the opposite segment.





Converse of Theorem 2: Angle between line and chord

- When a line drawn through the endpoint of a chord makes an angle with the chord that is equal to an angle on the circle subtended by the chord in the opposite segment of the circle, then the line is a tangent to the circle.

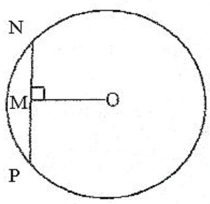


Questions

Question 1

In circle O, OM⊥NP, NP = 23,5 cm and OM = 13,5 cm.

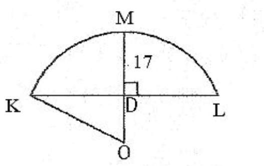
Calculate the length of:



1.1 NM 1.2 OP 1.3 the diameter of the circle.

Question 2

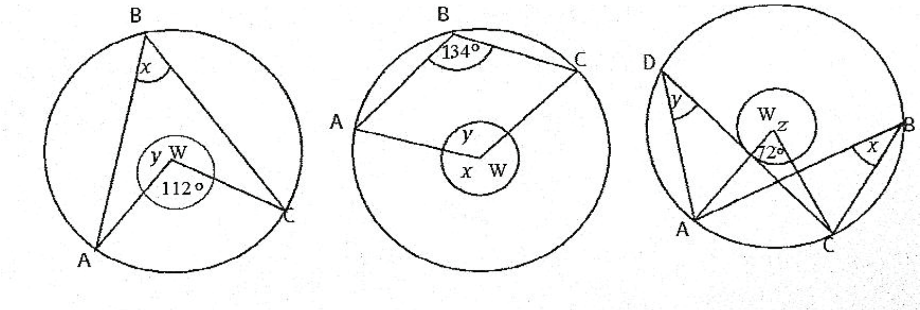
In the diagram, KL is the span of a bridge that forms the arc of a circle. KL is 103 m and the highest height of the bridge is 17 m.



Calculate the radius of the circle of which KML is an arc.

Question 3

Look at the diagram below.

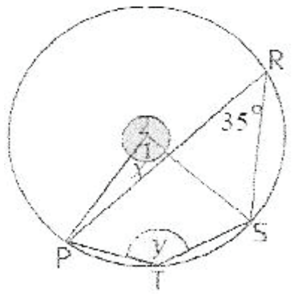


In circle W, write down the sizes of the angles marked by x, y and z.

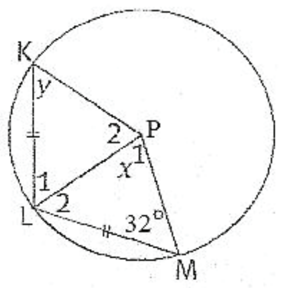
Question 4

Determine, with reasons, the values of x and y. In 4.2 P is the centre of the circle.

4.1

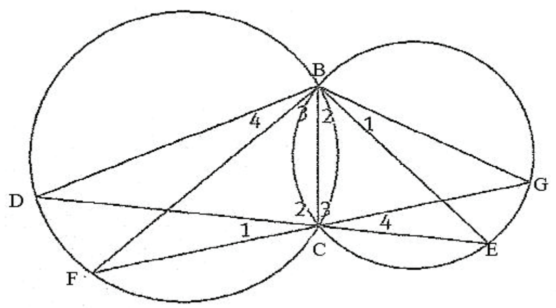


4.2



Question 5

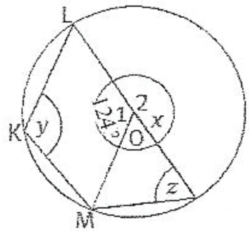
BC is a common chord to the circles intersecting at B and C. DCE and FCG are line segments intersecting at C.



Prove that $\widehat{B_4} = \widehat{B_1}$. Give reasons for all the steps in your proof.

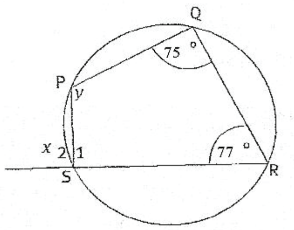
Question 6

Determine the values of the variables.



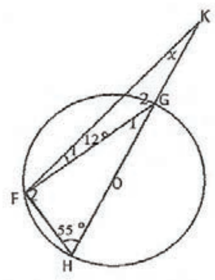
Question 7

Determine the values of x and y.



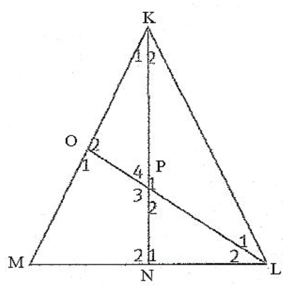
Question 8

Determine the value of x.



Question 9

In the diagram, KN⊥ML and LO⊥KM.



Prove that:

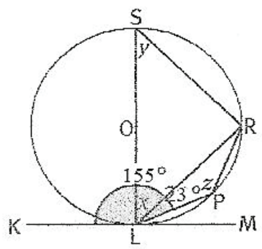
9.1 OPNM is cyclic

9.2 $K_1 = L_2$

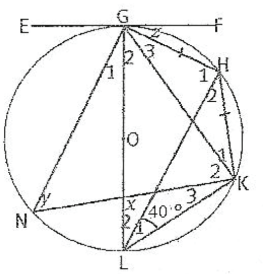
Question 10

O is the centre of the circle. KM, EF and AC are tangents. Determine, with reasons, the values of x, y and z.

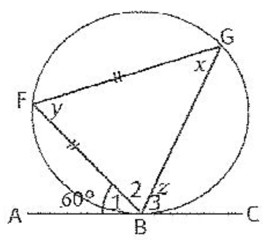
10.1



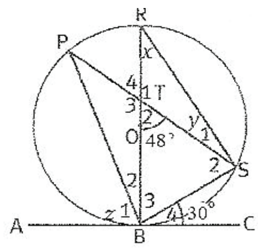
10.2



10.3

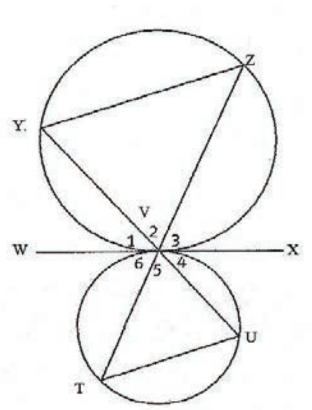


10.4



Question 11

Two circles touch externally at V. WVX is a tangent to both circles at V.



Prove that YZ||TU. Give reasons for all the steps in your proof.

Chapter 9

Trigonometry (area, sine , cosine rules)

Overview

Chapter 9 Page 94 Trigonometry (area, sine, cosine rules)	**Unit 1 Page 95**	
	The area rule	• Solving problems using the area rule
	Unit 2 Page 97	
	The sine rule	• Solving problems using the sine rule
	Unit 3 Page 99	
	The cosine rule	• Solving problems using the cosine rule
	Unit 4 Page 101	
	Solving problems in two dimensions	• Problems with numerical values • Problems with variables

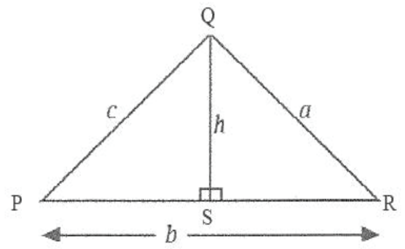
In this chapter, you will learn how to calculate the area of any triangle by using the area rule. You will also learn how to calculate different lengths and sizes by using the sine and cosine rules. Lastly, you will learn how to solve real-life problems including calculating heights, distances and angles.

The area rule

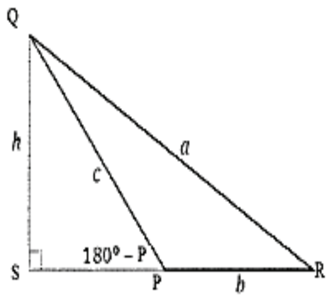
1.1 Solving problems using the area rule

- The area rule states that the area of any triangle is equal to half the product of any two sides, and the sine of the angle between the two sides.
- Proving the area rule:
  Consider ΔPQR in the following diagrams.

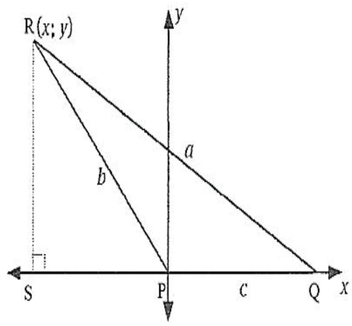
  $\hat{P}$ an acute angle:
  The area of ΔPQR $= (\frac{1}{2})bh$ ……(1)
  But, in ΔPQS, $\frac{h}{c} = \sin P \therefore h = c \sin P$
  Now, substitute this into (1):
  Area ΔPQR $= (\frac{1}{2})bc \sin P$



  $\hat{P}$ an obtuse angle:
  The area of ΔPQR $= (\frac{1}{2})bh$ ……(1)
  But, in ΔPQS, $\frac{h}{c} = \sin(180° - P)$
  $\therefore h = c \sin P$
  Now, substitute this into (1):
  Area ΔPQR $= (\frac{1}{2})bc \sin P$

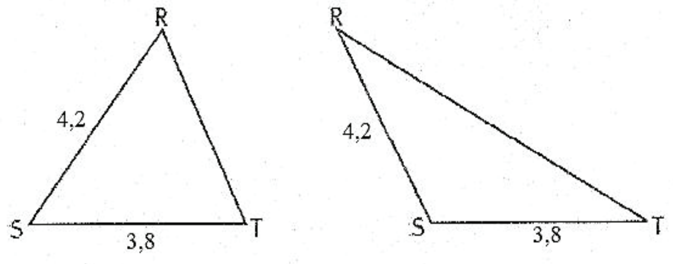


- The area rule can also be proved by using coordinates on the Cartesian plane:
  Let $\hat{P}$ be in the standard position. The coordinates of the apex R are ($b \cos P$; $b \sin P$).
  Let (x; y) be the coordinates of R.
  Take PQ to be the base and the y-coordinate of R as the perpendicular height. Then:
  $\frac{x}{b} = \sin P \therefore x = b \sin P$, and
  $\frac{y}{b} = \cos P \therefore y = b \cos P$
  The area ΔPQR $= (\frac{1}{2}) \times \text{base} \times \text{height}$
  $= (\frac{1}{2}).PQ.y_B$
  $= (\frac{1}{2})\, cb \sin P$



Example 1

Given Δ RST with RS = 4,2 cm and ST = 3,8 cm. If the area of Δ RST equals 7,55 cm^3, determine two possible sizes of $\hat{S}$.



$$\text{Area } \Delta \text{ RST} = \frac{1}{2} \,.\, \text{RS}.\, \text{ST}.\sin S$$

$$7{,}55 = \tfrac{1}{2}(4{,}2)(3{,}8).\sin S$$

$$\sin S = 0{,}946115288$$

$$\hat{S} = 71{,}11° \text{ or } \hat{S} = 180° - 71{,}11°$$

$$= 108{,}89°$$

The sine rule

2.1 Solving problems using the sine rule

- The sine rule states that, in any ΔPQR, $\frac{\sin P}{a} = \frac{\sin Q}{b} = \frac{\sin R}{c}$.
- It allows us to equate the ratios of the sine of the angles to their opposite sides.
- So, if we are given an angle and an opposite side, we can use the sine rule to solve the problem.
- Proving the sine rule:
  Consider ΔPQR in the following diagrams.

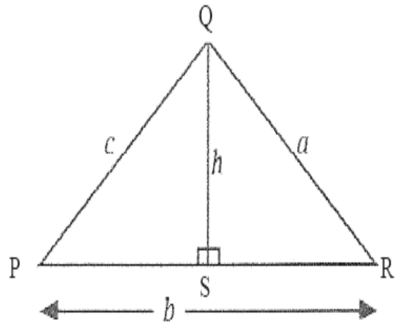
  $\hat{P}$ an acute angle:

  In ΔQPS: $\frac{h}{c} = \sin P \ \therefore h = c \sin P$

  In ΔQRS: $\frac{h}{a} = \sin R \ \therefore h = a \sin R$

  $\therefore c \sin P = a \sin R$

  Dividing both sides by ac yields
  $\frac{\sin P}{a} = \frac{\sin R}{c}$



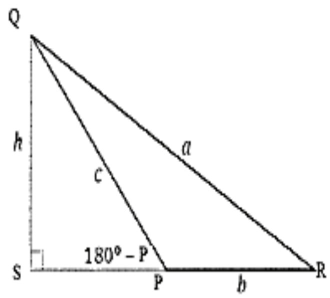
  $\hat{P}$ an obtuse angle:

  In ΔPQS: $\frac{h}{c} = \sin (180° - P) \ \therefore h = c \sin P$

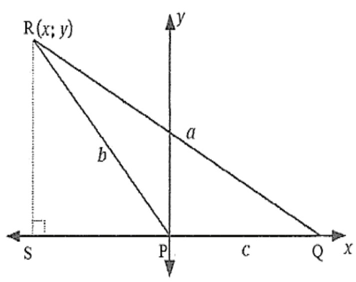
  In ΔQRS: $\frac{h}{a} = \sin R \ \therefore h = a \sin R$

  $\therefore c \sin P = a \sin R$

  Dividing both sides by ac yields
  $\frac{\sin P}{a} = \frac{\sin R}{c}$



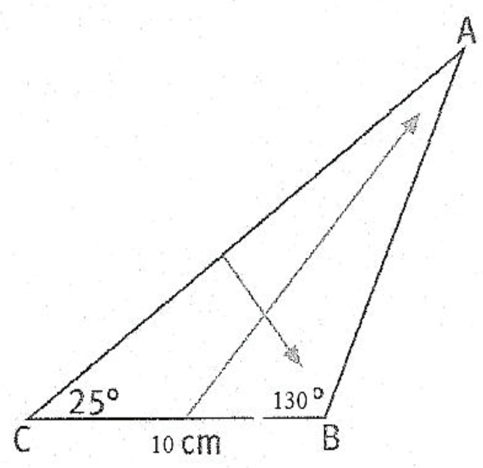
- The sine rule can also be proved by using coordinates on the Cartesian plane:
  Let $\hat{P}$ be in the standard position. The coordinates of R will be ($b \cos P$; $b \sin P$).
  If the y-axis is drawn through R, then the coordinates of R are ($a \cos Q$; $a \sin Q$)
  Now SR = $b \sin P = a \sin Q$
  Dividing both sides by ab yields
  $\frac{\sin P}{a} = \frac{\sin Q}{b}$



Example 2

1 Given two angles and one side of a triangle.

Determine the length of AC in ΔABC if BC = 10 cm, $\hat{B} = 130°$ and $\hat{C} = 25°$, correct to two decimal places.



$\hat{A} = 180° - (130° + 25°) = 25°$

Now we use the sine rule to calculate the length of AC.

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

Substitute the values into the rule:

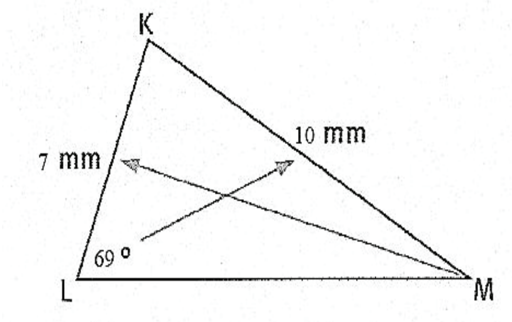
$$\therefore \frac{b}{\sin 130°} = \frac{10}{\sin 25°}$$

$\therefore b = AC = (10 \sin 130°)/\sin 25°$

$= 18{,}13 \text{ cm}$

2 Given two sides and an angle that is not included.

Solve ΔKLM if LK = 7 mm, KM = 10 mm and $\hat{L} = 69°$.



We can use the sine rule to determine $\hat{M}$:

$$\frac{\sin M}{7} = \frac{\sin 69°}{10}$$

$\therefore \sin M = 7 \sin 69°/10 = 0{,}653506298$

$\therefore \hat{M} = 40{,}81°$ or $\hat{M} \neq 180° - 40{,}81°$

$= 139{,}19°$

$\therefore \hat{M} = 40{,}81°$

Next we calculate the size of $\hat{K}$:

$\hat{K} = 180° - (69° + 40{,}81°) = 70{,}19°$

Finally we use the two angles ($\hat{L}$ and $\hat{K}$) and KM to determine the length of LM:

$$\frac{k}{\sin 70{,}19°} = \frac{10}{\sin 69°}$$

$\therefore k = 10 \sin 70{,}19°/\sin 69°$

$= 10{,}08 \text{ cm}$

3.1 Solving problems using the cosine rule

- The cosine rule states that, in any ΔPQR:
  - $a^2 = b^2 + c^2 - 2bc \cos P$ or $\cos P = (b^2 + c^2 - a^2)/2bc$
  - $b^2 = a^2 + c^2 - 2ac \cos Q$ or $\cos Q = (a^2 + c^2 - b^2)/2ac$
  - $c^2 = a^2 + b^2 - 2ab \cos R$ or $\cos R = (a^2 + b^2 - c^2)/2ab$
- So, if we are given 2 sides and the included angle, or three sides of a triangle, we can use the cosine rule to solve the problem.
- Proving the cosine rule:
  Consider ΔPQR in the following diagrams.

$\hat{P}$ an acute angle:

In ΔQSR: $a^2 = QS^2 + RS^2$

$= QS^2 + (b - PS)^2$

$= QS^2 + b^2 - 2bPS + PS^2$

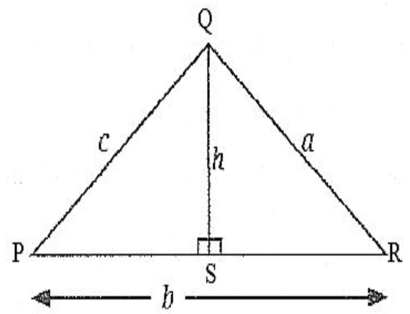
But $QS^2 + PS^2 = c^2$

$\therefore a^2 = b + c^2 - 2bPS$

In ΔPQS: $\frac{PS}{c} = \cos P$

$\therefore a^2 = b^2 + c^2 - 2bc \cos P$

or $\cos P = (b^2 + c^2 - a^2)/2bc$



$\hat{A}$ an obtuse angle:

In ΔQPR: $a^2 = QS^2 + PR^2$

$= QS^2 + (b - PS)^2$

$= QS^2 + b^2 - 2bPS + PS^2$

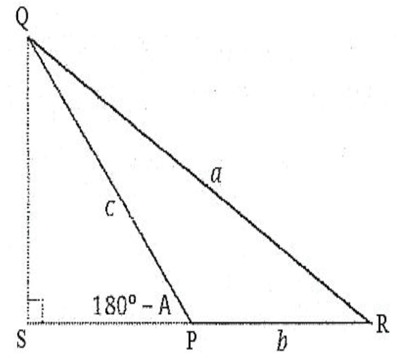
But $QS^2 + PS^2 = c^2$

$\therefore a^2 = b^2 + c^2 - 2bPS$

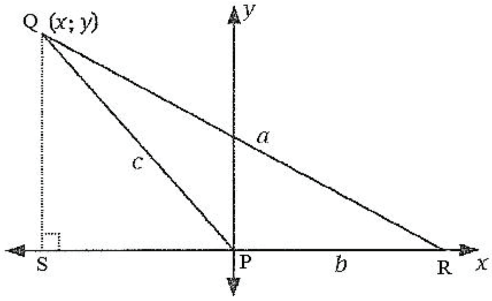
In ΔPQS: $\frac{PS}{c} = \cos(180° - A) = -\cos A$

$\therefore a^2 = b^2 + c^2 - 2bc \cos A$

Or $\cos A = (b^2 + c^2 - a^2)/2bc$

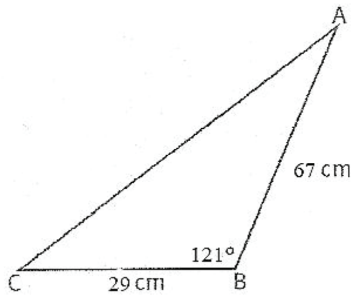


- The cosine rule can also be proved by using coordinates on the Cartesian plane: Let $\widehat{P}$ be in the standard position. The coordinates of R will be (b cos P; b sin P) and B(a; 0).
Using the distance formula,
$a = \sqrt{(b^2 - 2bc \cos P) + c^2(\cos^2 P + \sin^2 P)}$
But, $\sin^2 P + \cos^2 P = 1$
$\therefore a^2 = b^2 + c^2 - 2bc \cos P$



Example 3

Solve ΔABC if BC = 29 cm, ^B = 121° and c = 67 cm. Give your answer correct to two decimal places.



$b^2 = a^2 + c^2 - 2ac \cos B$

$= 29^2 + 67^2 - 2(29)(67) \cos 121°$

$= 7\ 331{,}437959$

$\therefore b = 85{,}62$ cm

Now we apply the sine rule to determine the magnitude of ^A:

sin A/29 = sin 121°/85,62

∴ sin A = 29 sin 121/85,62 = 0,29032763

∴ ^A = 16,88° or ^A ≠ 180° - 16,88° = 163,12°

∴ ^A = 16,88°

And so, ^C = 180° - (121° + 16,88°) = 42,12°

Solving problems in two dimensions

4.1 Problems with numerical values

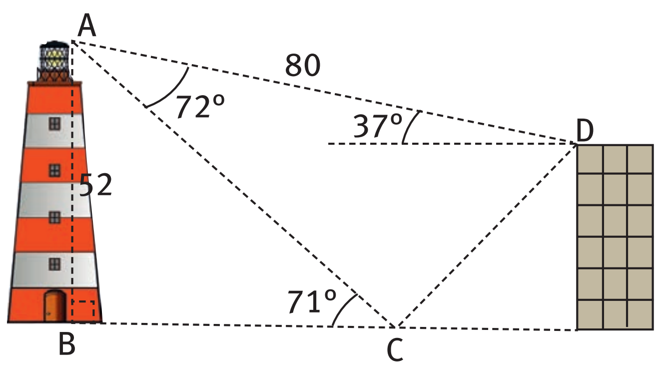
- When we know the angles of elevation between objects on the ground and higher objects, we can calculate the distances between the objects and the higher objects, and also the area between them, by using the area, sine and cosine rules.

4.2 Problems with variables

- When we know the angles of elevation, angles and area between objects on the ground and higher objects, we can use the area, sine and cosine rules to solve unknown variables in the data.

Example 4

1 The angle of elevation from a point C to the top of a lighthouse is 71°. The angle of elevation $\hat{D}$, from the top of an office block to the top of the lighthouse is 37°. AD = 80 m, the height of the lighthouse is 52 m, and $\widehat{CAD} = 72°$.



Calculate, correct to two decimal places:

1.1 the distance AC:

In ΔABC: $\frac{52}{AC} = \sin 71°$

$\therefore AC = {}^{52}/_{\sin 71°}$

$\approx 55{,}00$ m

1.2 the distance CD:

In ΔACD, $CD^2 = AC^2 + AD^2 - 2AC.AD.\cos \widehat{CAD}$

$= 3\,025 + 6\,400 - 2(55)(80)\cos 72°$

$= 6\,705{,}65045$

$\therefore CD = 81{,}89$ m

1.3 the size of $\widehat{CDE}$:

$\sin \widehat{ADC} \;/_{55,00} = \sin 72° /_{81,89}$

$\sin \widehat{ADC} = 0{,}638760635$

$\therefore \widehat{ADC} = 39{,}70°$

$\therefore \widehat{CDE} = 2{,}70°$

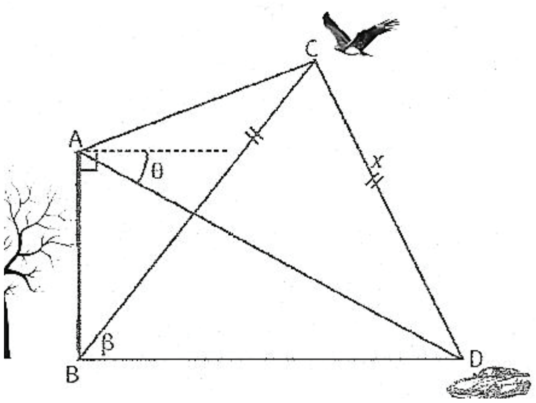
1.4 the area of ΔACD:

Area $\Delta ACD = \frac{1}{2} AC.AD.\sin \widehat{CAD}$

$= \frac{1}{2}(55)(80)\sin 72°$

$= 2\,092{,}32$ square units

2 A stargazer, A, climbs up a tall tree. She can see her car, D, parked a few metres from the tree. The angle of depression of D from A is θ. An eagle, C, flying past is at a specific time, equidistant from the foot of the tree and the car. At that time, the angle of elevation of C from B is ß. Let BC = CD = x meters.



Prove that $AB = \sqrt{2\,x \tan \theta}\,\sqrt{(1 + \cos 2ß)}$.

In $\Delta ABD, \widehat{ADB} = \theta$.

$AB/BD = \tan\theta$

$AB = DB \tan\theta$ ………. (1)

In ΔBCD:

$BD^2 = BC^2 + CD^2 - 2BC.CD.\cos\widehat{BCD}$

$= x^2 + x^2 - 2(x)(x)\cos(180° - 2ß)$

$= 2x^2(1 + \cos 2ß)$

$\therefore BD = \sqrt{2}.x.\sqrt{(1 + \cos 2ß)}$ ………. (2)

Substitute (2) in (1):

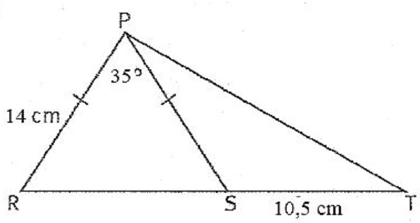
$AB = \sqrt{2}.x.\tan\theta.\sqrt{(1 + \cos 2ß)}$

Questions

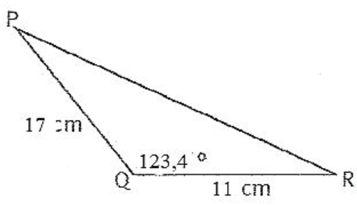
Question 1

Determine the area of each of the triangles, correct to two decimal places.

1.1



1.2



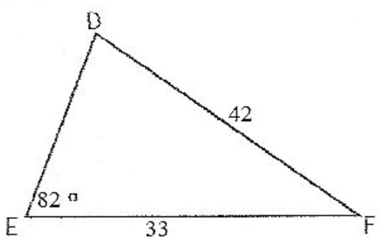
Question 2

ABC is a right-angled triangle. D is a point on AC such that DC = 24 units. If BC = 19 units and the area of ΔBCD = 109 square units, determine the length of AB correct to two decimal places.

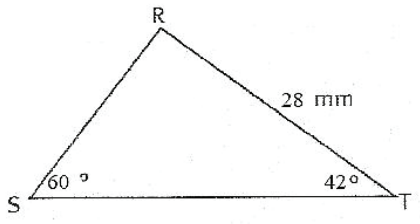
Question 3

Solve the following triangles, and give your answers correct to two decimal places.

3.1



3.2



Question 4

In the diagram below, HL = 21,3 cm, EF = 13,9 cm, $\widehat{HEF} = 128°$, $\widehat{HFE} = 37°$ and $\widehat{EGF} = 15°$. Determine, correct to two decimal units, the:

4.1 length of EF

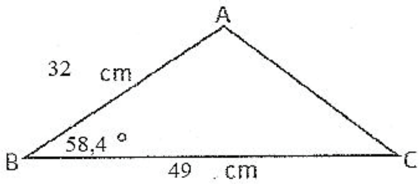
4.2 area of ΔHEF

4.3 size of $\widehat{HFG}$ if $\widehat{EFG}$ is an obtuse angle.

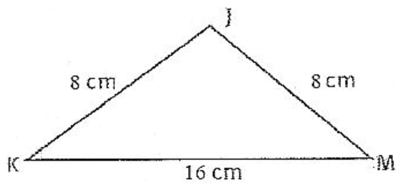
Question 5

Solve the following triangles, and give your answers correct to two decimal places.

5.1



5.2



Question 6

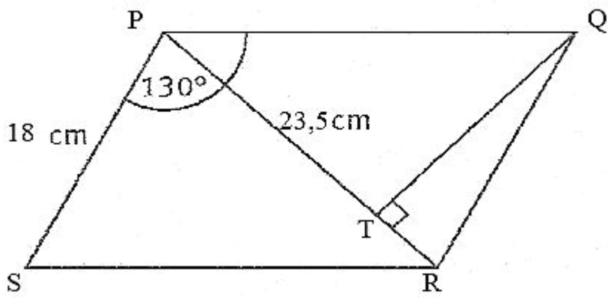
Prove that in any triangle VWX, $\cos W = (v^2 + x^2 - w^2)/2vx$.

Question 7

Calculate without using a calculator, the length of AC in ΔABC if BC = 12,5 cm, AB = 93 mm and $\widehat{B} = 124,7°$.

Question 8

PQRS is a parallelogram with PS = 18 cm, PR = 23,5 cm and $\hat{P} = 130°$. QT is drawn perpendicular to PR.



Calculate:

8.1 $P\hat{S}R$

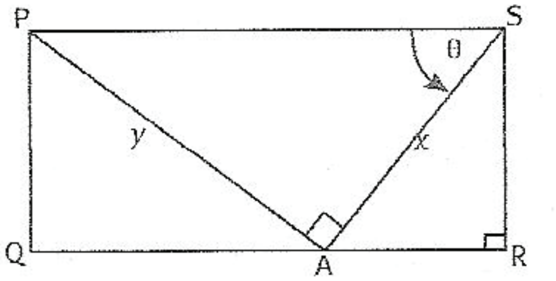
8.2 $P\hat{R}S$ (to two decimal places)

8.3 the area of ΔPSR (to the nearest square cm)

8.4 the length of QT.

Question 9

PQRS is a rectangle.



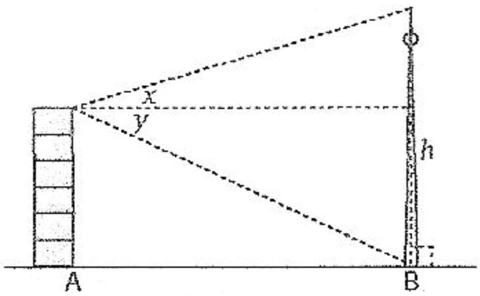
Express the following in terms of x and/or y and θ:

9.1 RS

9.2 AR

Question 10

The angle of elevation from the top of a suburban house to the top of a nearby apartment block is $x°$, and the angle of depression to the foot of the apartment block is $y°$. It is assumed that the height of the apartment building is 15 m, and the house and the apartment building lie in the same horizontal plane.

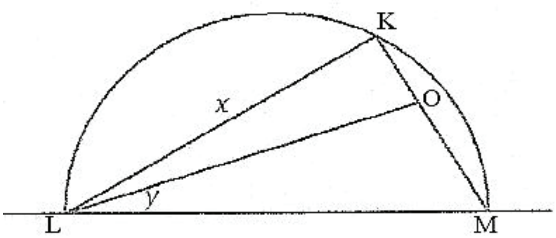


10.1 Show that AB, the distance between the house and the apartment block, is given by $\text{AB} = (15 \cos x . \cos y)/\sin(x + y)$.

10.2 If $x = 102°$ and $y = 30°$, what is the correct value of h, the height of the apartment block?

Question 11

LM is the diameter of a semi-circle. K is a point on the circumference. KM = 11,14 units. O is a point on KM such that KO:OM = 1:3. Let KL = x units and $\widehat{\text{OLM}} = \text{y}$.



Find the general solution for $\cos y$.

Chapter 10

Finance, growth and decay

Overview

Chapter 10 Page 107 Finance, growth and decay	**Unit 1 Page 108**	
	Compound growth	• Different compounding periods • Timelines • Nominal and effective interest rates
	Unit 2 Page 111	
	Decay	• Linear depreciation • The reducing balance method • Compound increase and decrease combined

In this chapter you will learn how to accurately and quickly calculate the present and future values of investments and debts, based on periodic payments at different interest rates, and compounded at different times.

Remember in Grade 10 you learnt about simple interest and compound interest.

Simple interest: $A = P(1 + i.n)$

Compound interest: $A = P(1 + i)^n$

where A is the final amount (principal plus interest)

P is the principal amount

i is the interest rate

n is the number of periods.

Remember, if the interest rate is 18%, we use 18/100 = 0,18 in the formula.

Compound growth

1.1 Different compounding periods

- When you take out a loan or invest money in an interest-bearing account, interest might be compounded once a year, or more often, depending on the contract entered into.
- An interest rate that is quoted yearly, but compounded more often, is called a nominal interest rate or quoted interest rate.
- If the interest rate is compounded n times per year and, let's say an interest rate of 13% is used, then the rate we use in the formula is 13%/n to include the number of compounding periods in our answer.
- More frequent compounding means a higher value of A, the final amount.

Example 1

Josephine invests R102 000 for three years. The bank quotes an interest rate of 12,5% p.a. and gives her a choice of different compounding periods:

- 12,5% compounded half-yearly
- 12,5% compounded quarterly
- 12,5% compounded monthly
- 12,5% compounded daily (use 365 days per year).

Determine the compounding period that gives Josephine the best return on her investment.

Let P be the initial amount invested (R102 000), let n be the number of compounding periods, and let i be the interest rate per compounding period.

Half-yearly: $A = 102\ 000(1 + \frac{0{,}125}{2})^{6} = \text{R}146\ 748{,}55$ ($n = 3$ years$\times 2$ periods $= 6$)

Quarterly: $A = 102\ 000(1 + \frac{0{,}125}{4})^{12} = \text{R}147\ 559{,}68$ ($n = 3$ years $\times$ 4 periods $= 12$)

Monthly: $A = 102\ 000(1 + \frac{0{,}125}{12})^{36} = \text{R}148\ 121{,}54$ ($n = 3\ years \times 12$ periods $= 36$)

Daily: $A = 102\ 000(1 + \frac{0{,}125}{365})^{1\ 095} = \text{R}148\ 399{,}60$ ($n = 3$ years $\times$ 365 periods $= 1\ 095$)

Therefore, the best return is given by daily compounding.

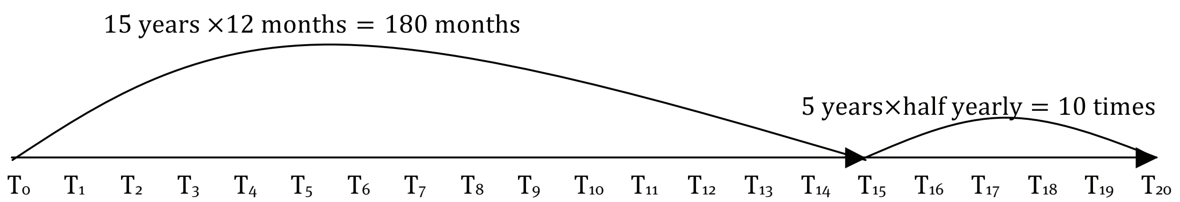
1.2 Timelines

- Timelines are used when we need to work with different compounding periods within a single investment or loan. It makes it easier to visualise the rates used over time.

Example 2

Musa takes out a loan of R500 000 for 20 years at an interest rate of 10%, compounded monthly for the first 15 years and at an interest rate of 18%, compounded half-yearly for the last 5 years.

We can show this graphically in the following way:



1.3 Nominal and effective interest rates

- When we calculate interest rates at the end of every compounding period, the actual interest rate earned or paid at the end of the investment or loan will not be the same as the quoted interest rate. This interest rate is called the effective interest rate, and is calculated by using the formula:

  $$i_{\text{eff}} = (1 + i_{\text{nom}}/m)^m - 1$$

  where i_{eff} is the effective interest rate
  i_{nom} is the nominal interest rate
  m is the number of compounding periods per year
- If the nominal interest rate is compounded annually, the effective interest rate will be equal to the nominal interest rate.

Example 3

Determine the effective annual interest rates for each of the following nominal interest rates, correct to two decimal places.

1 1.1 13,25% compounded half-yearly:

$$i_{\text{eff}} = (1 + \frac{0,1325}{2})^2 - 1$$

$$= 0,136889062 \text{ or } 13,69\%$$

1.2 13,25% compounded quarterly:

$$i_{\text{eff}} = (1 + \frac{0,1325}{4})^4 - 1$$

$$= 0,139230185 \text{ or } 13,92\%$$

1.3 13,25% compounded weekly:

$$i_{\text{eff}} = (1 + \frac{0,1325}{52})^{52} - 1$$

$$= 0,141486632 \text{ or } 14,15\%$$

1.4 13,25% compounded daily:

$$i_{\text{eff}} = (1 + \frac{0,1325}{365})^{365} - 1$$

$$= 0,141651566 \text{ or } 14,17\%$$

2 Which one of the nominal rates yields the highest return?

The rate of 13,25% compounded daily offers the best return.

Decay

Financial decay refers to the gradual loss of value. When an asset gradually loses value, it is called depreciation.

2.1 Linear depreciation

- Linear depreciation is calculated as a percentage of the original value. The same amount is deducted from the principal amount every compounding period.
- The formula used to calculate linear depreciation is:
  $A = P(1 - in)$

  where P is the original value of the asset
  A is the final value of the asset
  i is the rate at which the asset depreciates per year
  n is the number of years over which the asset depreciates.

Example 4

A tractor was bought in March 2008 for R198 000. The value of the tractor depreciates at a rate of 5% per year on a linear basis. Determine the value of the tractor after seven years.

The tractor depreciates by 5% per year: R198 000×5% = R9 900.

After seven years the tractor will be worth R198 000 − (7×R9 900) = R128 700.

Using the formula: $A = 198\ 000(1 - 0{,}05(7)) =$ R128 700.

2.2 The reducing balance method

- The reducing balance method depreciates an asset by calculating the depreciation on the reduced value of the asset at the end of every compounding period.
- The formula used to calculate depreciation by the reducing balance method is:
  $A = P(1 - i)^n$

  where P is the original value of the asset
  A is the final value of the asset
  i is the rate at which the asset depreciates per year
  n is the number of years over which the asset depreciates.

Example 5

A tractor was bought in March 2008 for R198 000. The value of the tractor depreciates at a rate of 5% per year on the reducing balance method. Determine the value of the tractor after seven years.

The tractor depreciates in value as follows:

Year 1: 5% of R198 000 = R9 900

R198 000 – R9 900 = R188 100

Year 2: 5% of R188 100 = R9 405

R188 100 – R9 405 = R178 695

Year 3: 5% of R178 695 = R8 934,75

R178 695 – R8 934,75 = R169 760,25

Year 4: 5% of R169 760,25 = R8 488,01

R169 760,25 – R8 488,01 = R161 272,24

Year 5: 5% of R161 272,24 = R8 063,61

R161 272,24 – R8 063,61 = R153 208,63

Year 6: 5% of R153 208,63 = R7 660,43

R153 208,63 – R7 660,43 = R145 548,20

Year 7: 5% of R145 548,20 = R7 277,41

R145 548,20 – R7 277,41 = R138 270,79

Therefore, after seven years the tractor will be worth R138 270,79.

Using the formula, $A = \text{R}198\,000(1 - 0{,}05)^7 = \text{R}138\,270{,}79$.

2.3 Compound increase and decrease combined

- In Grade 10 you learnt that inflation causes the prices of goods and services to increase over time. In the previous section you learnt how depreciation causes the value of assets to decrease over time. What happens when you are asked to calculate the effects of both in one question? It is best explained by an example.

Example 6

A hospital purchases a brand new ambulance for R899 995. The ambulance depreciates at 12% per annum on the straight line method The expected inflation rate over the next ten years is 6,1% per year. This means that the price of new ambulances will increase over the next ten years.

Calculate:

1 1.1 The scrap value of the ambulance after seven years:

$P = 899\,995, \quad n = 7, \quad i = 12\%$

$A = P(1 - i)^n$

$= 899\,995(1 - 0{,}12)^7 \qquad = 367\,805{,}99$

1.2 The cost of a new ambulance after seven years:

$P = 899\,995, \quad n = 7, \qquad i = 12\%$

$A = P(1 + i)^n$

$= 899\,995(1 + 0{,}12)^7 \qquad = 1\,989\,602{,}21$

1.3 The difference between the price of a new ambulance and the scrap value of the existing ambulance after seven years.

$1\,989\,602{,}21 - 367\,805{,}99 = 1\,621\,796{,}22$

2 Calculate how much the hospital must invest now, at an interest rate of 5,5% per annum, compounded monthly, to be able to afford a new ambulance after seven years.

$A = 1\,621\,796{,}22, \qquad n = 7, \qquad i = 0{,}055$

$A = P(1 + i)^n$

$1\,621\,796{,}22 = \qquad P(1 + \frac{0{,}055}{23})^7 \times 12$

$P = 1\,621\,796{,}22/(1 + \frac{0{,}055}{12})84$

$= 1\,104\,523{,}39$

Questions

Question 1

Which of the following investments yields the most after seven years?

1.1 R61 000 invested at 15% p.a. compounded every four months

1.2 R61 000 invested at 14% p.a. compounded daily.

Question 2

An investment yields R999 822 after ten years. When the investment was made ten years ago, the bank offered 3,2% p.a. compounded monthly. Calculate the initial investment amount.

Question 3

Dikeledi wants an invested amount to quadruple (be four times as much) in value within four years. At what interest rate, compounded quarterly, must she invest her money? Do you think this is possible?

Question 4

Stephni deposited Rx into an investment account nine years ago. During the first two years, the investment earned 12% interest p.a. compounded monthly. After two years, Stephni withdrew R19 201,85. She then re-invested the money at 7,5% p.a. compounded daily. After seven years, she receives a lump sum of R3 111 052.

Solve for x, correct to the nearest cent.

Question 5

The output of an investment that Larisse took out nineteen years ago is R5 825 156. During the first six years, the interest earned was 9% p.a. compounded yearly. During the last thirteen years, the investment earned 12,4% p.a., compounded monthly.

Calculate the initial investment.

Question 6

Sam deposits R30 125 into an investment account for five years. For the first two years, the investment earns 13% interest p.a. compounded monthly, and is worth R39 015,43. At the end of the five-year investment period, Sam receives a pay out of R47 540,04.

Calculate the interest earned in the last three years of the investment, compounded quarterly.

Question 7

Mr Mhlongo deposited R10 000 into a savings account at 4,32% p.a. compounded quarterly, for 15 years.

7.1 Use the nominal rate to calculate the final value of the investment after 15 years.

7.2 Determine the effective equivalent annual rate.

7.3 Determine if the effective annual rate gives the same final value for the investment.

Question 8

An NGO receives a donation of 1 650 kg of maize seed.

8.1 Determine for how many days they will be able to give seeds to poor communities, if they give seeds out at 3% per day on the straight line method. Round off your answer to two decimal places.

8.2 After how many days will they be left with 132 kg of maize seed if they give seeds out at 8% per day on the straight line method? Round off answer to two decimal places.

8.3 A concerned community member asks them if they can reduce their donations so that the seeds last for 75 days.

Calculate what percentage they can give out per day, correct to two decimal places.

Question 9

A doctor buys new medical equipment for his practice, valued at R2 605 252.

9.1 Determine the depreciated value of the equipment after seven years if the rate of depreciation is 12,5% p.a. on a reducing balance.

9.2 Determine the rate of depreciation on a reducing balance if the value of the equipment falls to R650 000 after four years.

Question 10

A plasma television screen that was bought four years ago is now worth R1 260. Calculate what was paid for the screen four years ago if the rate of depreciation on a reducing balance is 15% p.a.

Question 11

A school buys a new bus for R1 250 000. They know that the bus will have to be replaced in five years. This kind of bus normally depreciates at 17% on a reducing balance. The expected average rate of inflation over the next four years is 6,02%.

11.1 Calculate:

- 11.1.1 the value of the bus after five years.
- 11.1.2 the cost of a new bus after five years.
- 11.1.3 the difference between the cost of a new bus and the value of the old bus after five years.

11.2 A well-to-do parent donates R75 000 to the school for a new bus at the end of the third year. If the school invests this donation at 12% per annum, compounded monthly, will they be able to cover the shortfall between the cost of the new bus and the resale value of the old bus at the end of the five years?

Question 12

Mr Gumede donates 15 new computers to his local community centre. The current value of the computers is R45 000, and computers normally depreciate at 10% on a reducing balance. The centre will have to replace the computers after 2,5 years. Inflation is expected to be 7,26% annually for the next four years.

If Ms Sithole, another wealthy community member, donates R8 000 to the community centre at the same time that Mr Gumede donates the computers, and the centre invests the money at 15% p.a. compounded weekly, will they have enough money available after 2,5 years to replace the computers? If not, how much will they still need?

Chapter 11

Probability

Overview

Chapter 11 Page 117 Probability	**Unit 1 Page 118** Combinations of events	• Venn diagrams • Mutually exclusive events • Complementary events
	Unit 2 Page 120 Dependent and independent events	• Rules of probability
	Unit 3 Page 121 Tree diagrams	• Tree diagrams for dependent events • Contingency tables

Probability is used to analyse random events. It gives us an indication of the answer to the question "Will the event occur?", so that we are able to make better informed decisions.

In this chapter you will learn how to use Venn diagrams, mutually exclusive events and complementary events to find the probability. You'll also learn how to use tree diagrams and contingency tables to make the calculation of probabilities easier.

Combinations of events

1.1 Venn diagrams

- Venn diagrams are a visual way of illustrating the connection between different sets of data.
- The sets are shown in circles that make it easy to calculate the probability.
- S is the sample space, and the space where the circles intersect is where combined events lie.
- Venn diagrams also help us understand concepts such as mutual exclusivity and complementary sets.

1.2 Mutually exclusive events

- Two events are mutually exclusive if there are no common elements between the two events. If A and B are mutually exclusive, then
  - o P(A and B) = 0
  - o P(A or B) = P(A) + P(B)

1.3 Complementary events

- Two events are complementary if they are mutually exclusive and P(A) + P(B) = 1

Example 1

1 We throw three coins.

1.1 What is the sample space?

S = {HHH; HHT; HTH; THH; HTT; THT; TTH; TTT}

1.2 What is the probability of getting:

1.2.1 three heads? 1/8

1.2.2 two tails and a head? 3/8

1.2.3 three tails and a head 0

1.2.4 a tail and two heads 3/8

1.2.5 a head, a tail and a head in this sequence? 1/8

2 A school uses three different-coloured books in their FET Phase classes, Red (R), Green (G) and Blue (B). Of the 400 learners, 150 used R, 265 used G and 135 used B. Some learners use more than one colour book. 160 used only G, 30 used only B, 30 used G and B and 25 used R and G.

2.1 Represent the data in a Venn diagram.

2.2 How many learners used:

2.2.1 none of the books?

2.2.2 all three books?

2.2.3 B and R only?

2.2.4 only R?

2.3 What is the probability that a learner selected at random would use:

2.3.1 none of the books?

2.3.2 all three books?

2.3.3 B and R only?

2.3.4 only R?

3 Let S be the set of natural numbers between 21 and 30, therefore S = {21, 22, 23, …, 30}. Let K be the set of prime numbers between 21 and 30, therefore K = {23, 29}. Let A be the set of non-prime numbers between 21 and 30, therefore A = {21, 22, 24, 25, 26, 27, 28, 30}.

We find that:

3.1 $P(K) = 2/10 = 1/5$

3.2 $P(A) = 8/10 = 4/5$

3.3 $P(K \text{ and } A) = 0$

3.4 $P(K) + (A) = 1/5 + 4/5 = 1$

Dependent and independent events

If the occurrence of one event does not influence the occurrence of another event, the events are independent.

If the occurrence of one event changes the probability that another event will occur, the events are dependent.

2.1 Rules of probability

- Multiplication rule (independent events):
  If P(A) and P(B) represent the probabilities that events A and B will occur, then the probability that both A and B will occur is P(A and B) = P(A) × P(B).
- Multiplication rule (dependent events):
  If event B is dependent on event A, then if B can still occur after event A, then the probability that both A and B will occur is P(A and B) = P(A) × P(B|A), where P(B|A) is the probability that B will occur, given that A has already occurred.
- Addition rule 1:
  If A and B are two mutually exclusive events and P(A) and P(B) represent the probability that each will occur, then the probability that either A or B will occur is P(A or B) = P(A) + P(B).
- Addition rule 2:
  If A and B are two events that are not mutually exclusive, and P(A) and P(B) represent the probability that each will occur, then the probability that either A or B will occur is P(A or B) = P(A) + P(B) – P(A) × P(B).

Example 2

1 Brenda first rolls a red die, and then she rolls a green die. What is the probability that she will get a 6 on the red die and a number less than four on the green die?

$P(6\text{ on red}) = 1/6$ and $P(\text{less than four on green}) = 3/6 = 1/2$

Therefore, $P(6\text{ on red and less than four on green}) = 1/6 \times 1/2 = 1/12$

2 Given events P and Q:

P(P) = 0,7 P(Q) = 0,2 P(P and Q) = 0,14

2.1 Are events P and Q mutually exclusive? Explain your answer.

No, P(P and Q) not equal 0.

2.2 Are events P and Q independent? Explain your answer.

Yes, P(P and Q) = P(P) × P(Q).

Tree diagrams

Tree diagrams are also known as probability trees as they show the probability of combinations of events. Each branch on the tree represents a single combination.

3.1 Tree diagrams for dependent events

- The difference between tree diagrams for dependent events and tree diagrams for independent events is that the probabilities for dependent events will change, because the probability of the second event is dependent on the first event.

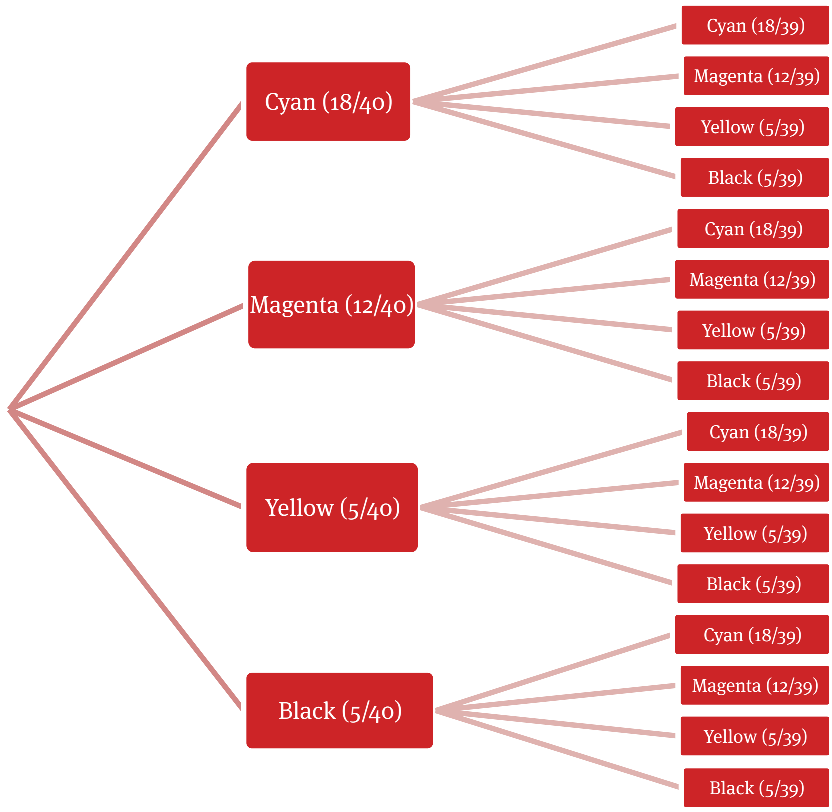
3.2 Contingency tables

- Contingency tables are used to set data out in a table. The data can be obtained from surveys of any form.
- Once the data is tabulated, it is easy to calculate if one variable is dependent on another in the survey.
- Contingency tables are especially useful when surveying the attitudes of people in a community towards something that has happened or is planned for the community.

Example 3

1 There are 40 coloured balls of identical size and shape in a bag. There are 18 cyan, 12 magenta, 5 yellow and 5 black balls. Two balls are drawn from the bag, one after the other, without replacing the first ball before the second is drawn.

1.1 Draw a tree diagram to illustrate the possible outcomes.



1.2 What is the probability of drawing the following balls? Round off your answers to two decimal places.

1.2.1 a cyan and then a magenta ball

P(cyan then magenta) = $18/40 \times 12/39 = 9/65 = 0{,}14$

1.2.2 two magenta balls

P(two magenta balls) = $12/40 \times 11/39 = 11/130 = 0{,}08$

1.2.3 a yellow and then a black ball

P(yellow then black) = $5/40 \times 5/39 = 5/312 = 0{,}02$

2 A group of elderly people were asked if they utilise the disabled parking bays at a shopping mall. Of the 400 respondents, 35 were disabled, and 365 were not disabled. The results of the questioning are tabled below:

	Disabled	**Not disabled**	**Total**
Used disabled parking bays	12	305	317
Did not use disabled parking bays	23	60	83
Total	35	365	400

Calculate the probability that a person drawn at random from the sample:

2.1 is disabled

P(is disabled) = 35/400 = 0,09

2.2 is a disabled person who uses the disabled parking bays

P(disabled and uses disabled parking) = 12/400 = 0,03

2.3 is not disabled and does not use the disabled parking bays

P(not disabled and does not use disabled parking bays) = 60/400 = 0,15

2.4 uses the disabled parking bays and is not disabled.

P(uses disabled parking bays) × P(is not disabled) = 317/400 × 365/400 = 0,72

Round off your answers to two decimal places.

Questions

Question 1

You throw a die and a coin.

1.1 What is the sample space?

1.2 What is the probability of getting:

1.2.1 an odd number and a head?

1.2.2 a prime number and a tail?

1.2.3 a number less than five and a head?

1.2.4 an even number and a tail?

Question 2

A group of 150 employees were asked if they use Microsoft Office Excel or OpenOffice.org Calc. Of the group, 72 said they use Excel, 78 said they use Calc, and 15 said they use neither.

2.1 Draw a Venn diagram to show this information.

2.2 How many employees used:

2.2.1 both programs?

2.2.2 only Calc?

2.2.3 only Excel?

2.2.4 both Excel and Calc?

2.3 What is the probability that an employee picked at random will use:

2.3.1 both programs?

2.3.2 only Calc?

2.3.3 only Excel?

2.3.4 both Excel and Calc?

Question 3

A five-sided die is thrown. What is the probability that:

3.1 the number will be less than 4?

3.2 the number will be greater than 4?

3.3 the number will be greater than or equal to four?

3.4 the number will be divisible by 3?

3.5 the number will not be divisible by 3?

3.6 What can you say about the sets in 3.4 and 3.5 and the sets in 3.1 and 3.3?

Question 4

A sports club asked their members which sports they prefer. There are 250 members in the club, of which:

180 like rugby (R)

95 like soccer (S)

40 like basketball (B)

70 like rugby and soccer

16 like all three sports

5 like soccer and basketball

235 like rugby or soccer or basketball

4.1 Draw a Venn diagram based on the given information.

4.2 How many members like none of the three sports?

4.3 How many members like basketball and soccer, but not rugby?

4.4 What is the probability that a member from the club drawn at random will like at least two of these sports?

Question 5

The probability that event A will occur is 2/5, the probability that event B will occur is 5/12 and the probability that event C will occur is 1/3. What is the probability that:

5.1 A or C will occur?

5.2 Neither A nor B will occur?

Question 6

The probability of rain in Port Louis, the capital of Mauritius, is 95% in February. The probability of a festival in dry weather is 32% during February, and one-and-a-half times more likely when it is wet.

6.1 Draw a tree diagram to illustrate all possible events and their probabilities.

6.2 What is the probability that:

6.2.1 there will be a festival?

6.2.2 there will not be a festival in wet weather?

Question 7

A survey was conducted on 4 895 people to determine whether drinking more than five cups of coffee per day was dependent on gender. Their responses are given in the table below.

	More than 5 cups	**Less than 5 cups**	**Total**
Female	(a)	1 565	(b)
Male	1 485	(c)	2 658
Total	2 157	(d)	4 895

7.1 Complete the table.

7.2 What is the probability that:

7.2.1 a female drinks more than five cups of coffee per day?

7.2.2 a person will be female?

7.2.3 a person will break a leg?

7.3 Determine whether drinking more than five cups of coffee per day is dependent on gender. Substantiate your answer with relevant calculations, rounded off to two decimal places.

Question 8

A car manufacturer offers two types of transmission in its entry-level model, automatic (AT) or manual (MT). Buyers can choose between petrol (P), diesel (D) or hybrid (H) engines, and the car comes in red (R), silver (S) or black (B).

8.1 Draw a tree diagram to show all the possible options.

8.2 What is the probability that a car drawn at random is:

8.2.1 black?

8.2.2 an automatic with a hybrid engine?

8.2.3 a manual with a diesel engine and red?

8.2.4 a petrol engine with silver paint?

Statistics

Overview

Statistics is the study of information. It is about making informed decisions based on analysis and interpretation of the data. Informed decisions are better decisions, and are especially helpful when investing money on the stock markets.

In this chapter you will learn about histograms, which are used to represent grouped data, frequency polygons, also used to represent grouped data, and ogives used to represent cumulative frequency. You will also explore the variance and standard deviations used to describe the spread of data, and learn to identify outliers.

Here is a quick recap of what you already know:

Ungrouped data	**Grouped data**
Measures of central tendency (mean, median, mode)	Measure of central tendency (estimated mean, modal class, class containing the median)
Measure of dispersion (range, quartiles, percentiles and deciles, interquartile range, semi-quartile range)	
Five-number summary	
The box-and-whisker plot	

Histograms

- A histogram is a graphical representation of grouped data. It consists of bars, like a bar graph, but unlike a bar graph, there are no gaps between the bars of a histogram.
- The width of a bar in a histogram is equal to the width of a class interval, and the height of a bar corresponds to frequency in that class interval.
- Class intervals are normally of the same length, with a lower and an upper limit, and the midpoint of the class interval is called the class mark.
- The **modal class** is represented by the highest bar.
- The **median** will fall in one of the class intervals.

How to draw a histogram:

1. Draw a set of axes with the class intervals marked out on the horizontal axis.
2. Mark equal intervals on the vertical axis, and make sure that the vertical axis can accommodate the highest frequency as read from the frequency table.
3. Draw the bars corresponding to the frequencies as read off the frequency table from the lower to the upper limit of the class interval, and shade the bar. Remember, the heights of the bars must correspond to the frequencies in the class intervals.

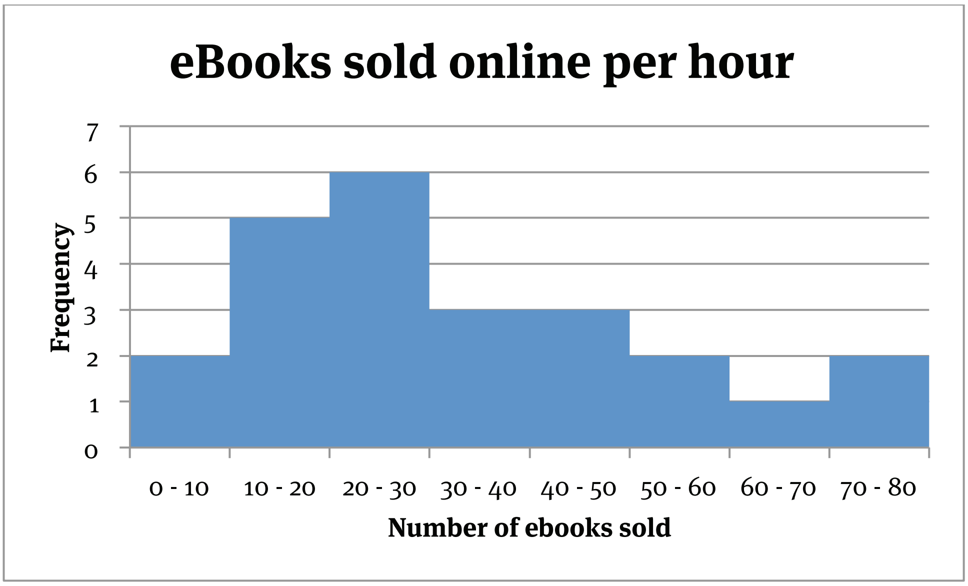
Example 1

The numbers of ebooks sold on an online store per hour over a 1-day period were recorded as follows: 2, 9, 14, 15, 16, 17, 19, 20, 20, 21, 24, 25, 27, 36, 37, 38, 45, 47, 48, 56, 58, 63, 72, 75.

The following frequency table represents this data:

eBooks sold per hour	$0 \leq x < 10$	$10 \leq x < 20$	$20 \leq x < 30$	$30 \leq x < 40$	$40 \leq x < 50$	$50 \leq x < 60$	$60 \leq x < 70$	$70 \leq x < 80$
Frequency	2	5	6	3	3	2	1	2

1 Draw a histogram to represent this data.



2 Determine the modal class.

The modal class is represented by the highest bar. Therefore the modal class is $20 \leq x < 30$.

3 Determine the class containing the median.

There are 24 data points in total. Therefore, the median is the average of the 12^{th} and 13^{th} values. This means that the 12^{th} and 13^{th} data values lie in the third interval, $20 \leq x < 30$.

Frequency polygons

A frequency polygon is a broken line graph used to represent grouped data. We draw a frequency polygon by starting with a histogram and joining all the class marks at the top of the bars by straight lines. Remember to ground your frequency polygon at each side of the histogram.

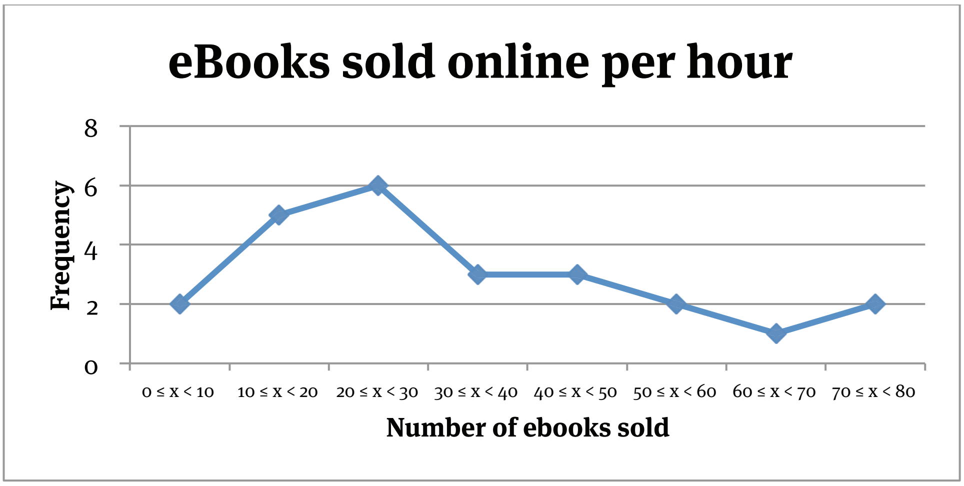
It is also possible to draw a frequency polygon without the use of a histogram. We do it in the following way:

1. Use the frequency table to calculate the class mark of every class interval.
2. Draw a set of axes for the data, and mark each of the frequencies corresponding to the class interval on it.
3. Join the marks to create the frequency polygon.
4. Remember to ground your frequency polygon on both sides.

Example 2

Draw a frequency polygon for the data set in Example 1 of Chapter 12.

eBooks sold per hour	**Class mark**	**Frequency**
$0 \leq x < 10$	5	2
$10 \leq x < 20$	15	5
$20 \leq x < 30$	25	6
$30 \leq x < 40$	35	3
$40 \leq x < 50$	45	3
$50 \leq x < 60$	55	2
$60 \leq x < 70$	65	1
$70 \leq x < 80$	75	2



Ogives

Ogives are graphs representing cumulative frequency, and as such are also known as cumulative frequency graphs.

The cumulative percentage is used on the vertical axis, and from that we can find useful measures like the median. We can also use an ogive to read quartiles, construct a box-and-whisker plot and read percentiles.

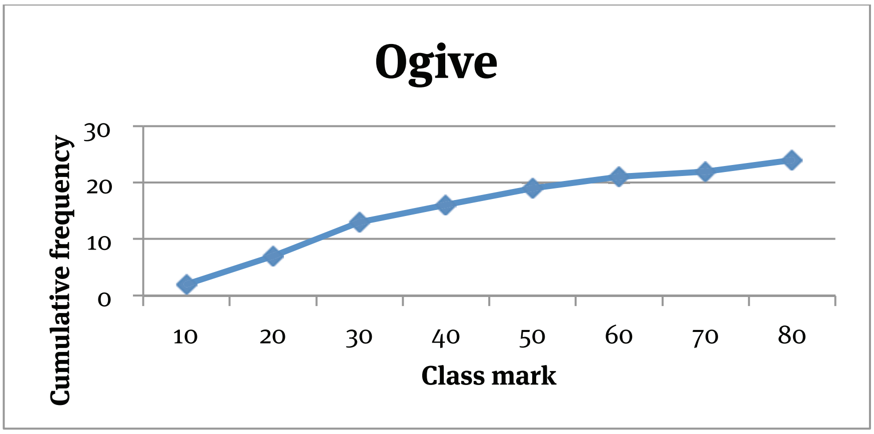
How to draw an ogive:

1. Use the frequency table given to construct a cumulative frequency table.
2. Calculate the coordinates of the points to plot: the upper limit of each interval to the cumulative frequency.
3. Join the points with a smooth curve, and ground the ogive by adding an interval to the left of the first one.

Example 3

Draw an ogive for the data set in Example 1 of Chapter 12.

Class intervals	**Frequency**	**Cumulative frequency**	**Points to plot**
$0 \leq x < 10$	2	2	(10; 2)
$10 \leq x < 20$	5	7	(20; 7)
$20 \leq x < 30$	6	13	(30; 13)
$30 \leq x < 40$	3	16	(40; 16)
$40 \leq x < 50$	3	19	(50; 19)
$50 \leq x < 60$	2	21	(60; 21)
$60 \leq x < 70$	1	22	(70; 22)
$70 \leq x < 80$	2	24	(80; 24)



Variance and standard deviation of ungrouped

When we need an accurate description of the spread of data, we look to see how the individual items deviate from the mean.

Because the differences add up to 0, we square the deviations and calculate the variance (s^2): $s^2 = (\Sigma(x_i - \bar{x})^2)/n$

This statistic is very useful, but we cannot use it directly as the units are not the same as the sample data. To correct this problem, we take the square root of the variance to calculate the standard deviation (s): $s = \sqrt{s^2}$

If the standard deviation is small, it means that the values are closely bunched around the mean. If the standard deviation is large, it means that the values are scattered away from the mean.

When a data set is normally distributed, we can say that:

- 68% of the values lie within one standard deviation from the mean,
- 95% of the values lie within two standard deviations from the mean, and
- 97,7% of the values lie within three standard deviations from the mean.

Example 4

Calculate the variance and standard deviation of the following data set:

15; 21; 1; 9; 5; 13; 19; 7

x_i	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$
15	3,75	14,0625
21	9,75	95,0625
1	-10,25	105,0625
9	-2,25	5,0625
5	-6,25	39,0625
13	1,75	3,0625
19	7,75	60,0625
7	-4,25	18,0625
n = 8, $\bar{x}$ = 90/8 = 11,25		Σ = 339,5

$$s^2 = \frac{339,5}{8} = 42,4375$$

$$s = \sqrt{s^2} = \sqrt{42,4375}$$

$$= 6,51$$

Using your calculator to calculate the standard deviation

Suppose we have the following small data set:

10, 12, 14, 15, 18, 21.

To calculate the standard deviation using a Sharp EL531WH (this is the calculator that most learners use, but the procedure is similar for other calculators):

Press	**You will see**
Mode 1 0	STAT 0
10 DATA	Data set 1
12 DATA	Data set 2
14 DATA	Data set 3
15 DATA	Data set 4
18 DATA	Data set 5
21 DATA	Data set 6
RCL n	6
RCL $\bar{x}$	$\bar{x} = 15$
RCL sx	$sx = 4$
RCL σx	$\sigma x = 3,6515$
RCL Σx	$\Sigma x = 90$

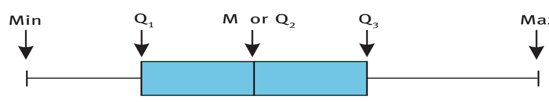
Note that sx is the standard deviation of a sample and σx is the standard deviation of a population.

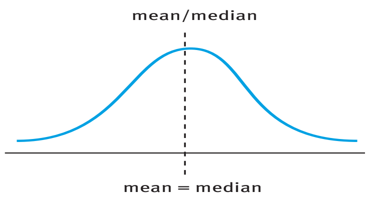
Symmetrical and skewed data

If a distribution is skewed, it means that one of its tails (from the median to an end-point) is longer than the other one. We can show this graphically in a box-and-whisker plot or a frequency polygon.

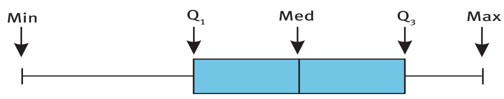
There are three different cases that you need to remember:

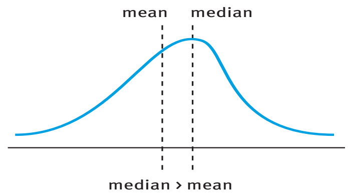
1 The data is symmetrical about the median. This means that the mean is equal to the median.



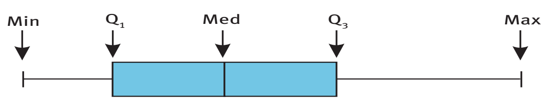


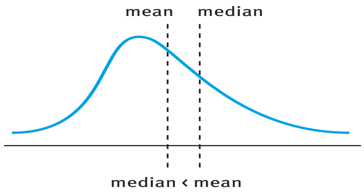
2 The data is skewed to the left, or negatively skewed. This means that the mean is smaller than the median.





3 The data is skewed to the right, or positively skewed. This means that the mean is greater than the median.



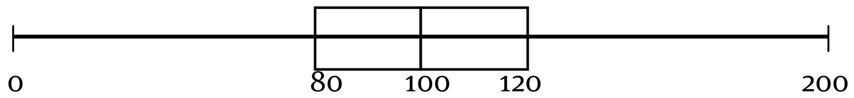


When data is shown graphically like this it is easier to read off the values that are needed to analyse the data.

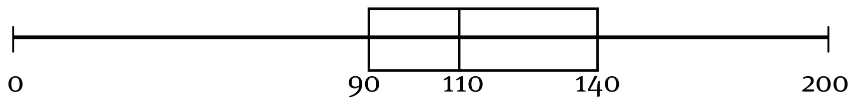
Example 5

The box-and-whisker plots of two sets of data (A and B) are shown in the following diagram with all the necessary data values.

Data set A:



Data set B:



1 Which data set is the most symmetrical?

Data set A is the most symmetrical.

2 Which data set has the larger interquartile range?

Data set B has the larger interquartile range.

3 For the data set that is skewed, note if it is skewed to the left or the right.

Data set B is skewed to the left.

Identifying outliers

- When collecting or analysing data, it is very easy for the person recording the data to make a mistake. A mistake can also creep in when capturing the data on computer for later analysis.
- Outliers are extreme values (they differ greatly from the rest of the values) that might be correct, or incorrect, caused by data capturing mistakes.
- We usually identify outliers by representing data graphically. The most commonly used way is to create a scatter plot.
- Scatter plots show the dispersion of bivariate data, and can take on any one of many forms. For our purposes, most scatter plots will have points forming a pattern (or trend) resembling a straight line. We can fit a 'line of best fit' through this data by drawing a straight line through as many points as possible, with (ideally) as many points above the line as below. The line of best fit can be used to predict future values for the data set.
- Box-and-whisker plots can also help us identify outliers. Here we define an outlier as any value greater (or less) than 1,5 times the interquartile range.

Example 6

Check the following data sets for outliers:

1 25, 28, 33, 35, 41, 62

Min = 25 Q_1 = 26,5 Median = (33 + 35)/2 = 34

Q_3 = 51,5 Max = 62

Interquartile range = 51,5 − 26, 5 = 25 1,5 × 25 = 37,5

Therefore, any reading below 26,5 − 37,5 = -11

or any reading above 51,5 + 37,5 = 89 is an outlier.

∴ This data set has no outliers.

2 11, 45, 48, 50, 51, 52, 53, 59, 62, 64, 69

Min = 11 Q_1 = 48 Median = 52

Q_3 = 62 Max = 69

Interquartile range = 62 − 48 = 14 1,5 × 14 = 21

Therefore, any reading below 48 − 21 = 27

or any reading above 62 + 21 = 83 is an outlier.

∴ 11 is an outlier in this data set.

3 1, 3, 5, 29, 32, 35, 38, 41, 89, 92, 96, 102

Min = 1 Q1 = 5 Median = 36,5

Q3 = 92 Max = 102

Interquartile range = 92 – 5 = 87 $1,5 \times 87 = 130,5$

Therefore, any reading below 5 – 130,5 = -125,5

or any reading above 102 + 130,5 = 232,5 is an outlier.

∴ This data set has no outliers.

Questions

Question 1

In an effort to increase the Mathematical Literacy levels amongst young adults, the Department of Education tested 50 twenty-year-olds' skills, sent them all on a course, and after the course tested their skills again. After the course their scores (in percentage terms) were:

15, 17, 19, 26, 28, 35, 39, 41, 41, 42, 45, 45, 45, 45, 47, 52, 53, 54, 57, 59, 55, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 68, 67, 72, 75, 74, 78, 81, 82, 82, 82, 82, 82, 82, 91, 95, 92, 99, 98, 94

1.1 Represent this data in a frequency table with intervals of 10.

1.2 Give the mode.

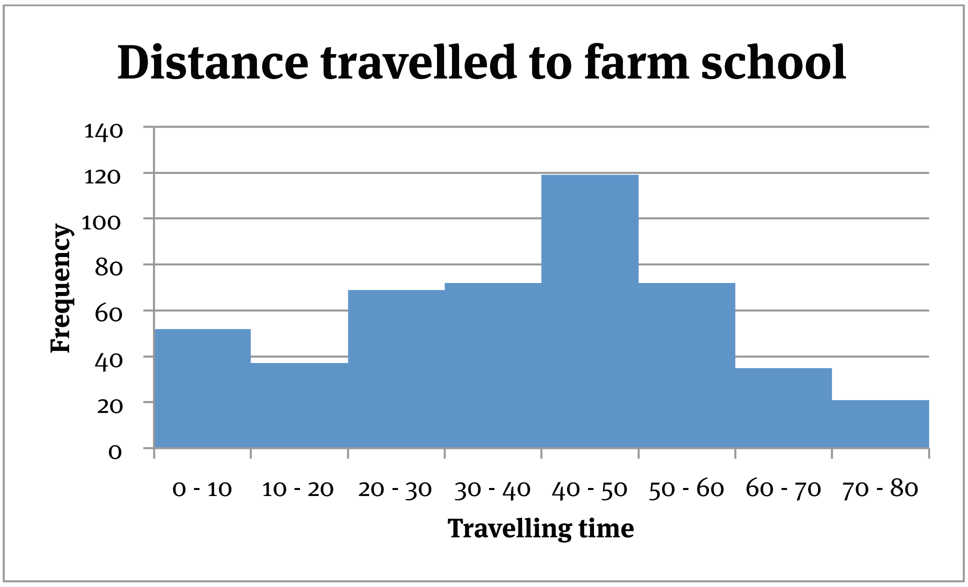
1.3 In which class will the median lie?

1.4 Draw a histogram of the data.

1.5 Construct a frequency polygon on the histogram.

Question 2

A survey was conducted at a small farm school to establish how far learners travelled to school daily. The histogram below represents the results of the survey:



2.1 Calculate the estimated mean travelling time by using a frequency table.

2.2 Give the modal class.

2.3 Draw a frequency polygon.

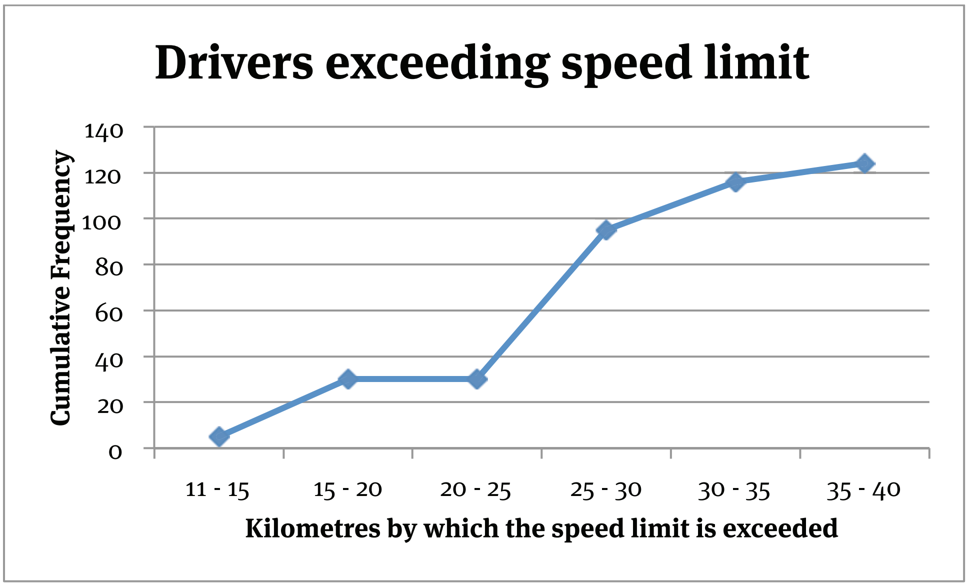
Question 3

Represent the following data in an ogive. Show how you would use the ogive to read off the median, lower and upper quartiles.

Interval	Frequency
$10 \leq x < 20$	23
$20 \leq x < 30$	37
$30 \leq x < 40$	45
$40 \leq x < 50$	88
$50 \leq x < 60$	52
$60 \leq x < 70$	16
$70 \leq x < 80$	12

Question 4

An ogive showing how many drivers exceed the speed limit, and by how many kilometres they exceed it, is given below.



4.1 How many drivers exceeded the speed limit?

4.2 How many drivers exceeded the speed limit by more than 20 km/h?

4.3 Construct a cumulative frequency table for the data given.

Question 5

Calculate the variance and standard deviation of the following data set:

155, 142, 169, 133, 189, 128, 175, 168, 135

Question 6

A survey was conducted on the weight of 10 000 adult female koi fish in breeder ponds. It was found that the average mass of an adult female koi was 7,5 kg with a standard deviation of 2 kg.

6.1 How many adult female koi weighed between 5,5 kg and 9,5 kg?

6.2 How many of the female koi weigh less than 3,5 kg?

6.3 What percentage of the koi weigh more than 15,5 kg?

6.4 Do any of the adult female koi weigh more than 20 kg? If so, how many do?

Question 7

The average hourly rate charged by a Mathematics editor is R280 with a standard deviation of R58, by a Business Studies editor R230 with a standard deviation of R23, and an Arts and Culture editor R180 with a standard deviation of R69. A Mathematics editor charges R350 per hour, a Business Studies editor R195, and an Arts and Culture editor R210. Which editor is relatively more expensive?

Question 8

A group of student bookkeepers is tested on the time it takes them to type 3 000 words. Their scores, in seconds, are given by:

125, 99, 134, 149, 138, 124, 133, 146, 117, 137

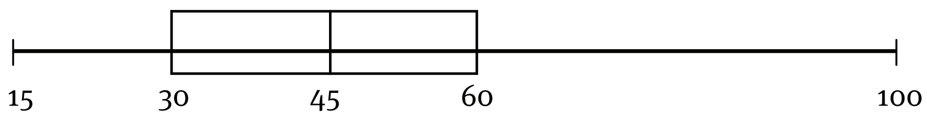
8.1 Calculate the mean time taken to type 3 000 words.

8.2 Calculate the standard deviation of the time taken to type 3 000 words.

8.3 How many of the student bookkeepers completed the 3 000 within two standard deviations of the mean?

Question 9

Consider the box-and-whisker plot below.



9.1 Write down the five-number summary.

9.2 Determine the semi-quartile range.

9.3 Comment on the spread of the data. What kind of data, do you think, might this represent?

9.4 Comment on the skewness of the data.

Question 10

Draw scatter plots for the following sets of pairs. Indicate any outliers.

10.1

x	3	2	5	1	4	6	8	5	4	5	4	5
y	1	2	3	2	1	2	3	2	1	2	3	2

10.2

x	4	2	5	8	1	2.5	5	6	8.5	2	9	4
y	1	1	1	0	0	0	3	7	2	9	5	0

10.3

x	1	5	2	3	6	4	5	6	2	3	1	2
y	3	6	9	5	6	9	3	5	6	9	6	5

Exam paper 1

MATHEMATICS PAPER 1

MARKS: 150 **TIME: 3 hours**

INSTRUCTIONS AND INFORMATION

Read the following instructions carefully before answering the question paper:

1 This question paper consists of 7 questions. Answer ALL questions.

2 Show ALL your calculations, diagrams, graphs, etcetera which you have used in answering the questions CLEARLY.

3 An approved scientific calculator (non-programmable and non-graphical) may be used unless stated otherwise.

4 Answers should be rounded off to TWO decimal places, unless stated otherwise.

5 Number your answers according to the numbering system used in this question paper.

6 Diagrams in the question paper are not necessarily drawn to scale.

7 It is in your own interest to write legibly (clearly) and to present your work neatly.

QUESTION 1

1.1 Solve for x in each of the following:

1.1.1 $9x(x - 6) = 15$

1.1.2 $x^2 - 4x \geq -4$

1.1.3 $1/2\ x^2 - 17x = 5$

1.2 Consider the equation: $1/(x - 9) + 2 = 3x/(9 - x)$

1.2.1 Why is $x \neq 9$?

1.2.2 Could $x = 3^2$?

1.2.3 Solve the equation.

1.3 Solve for both x and y in the systems below:

1.3.1 $2x + 2y = 2$ and $2x^2 + 4y^2 = 4$

1.3.2 $4x^2 + 6y - 5 = 0$ and $2x^2 - 7y = 15y$

1.4 Sibusiso keeps chickens in a rectangular chicken coup in his parents' back yard. The coup has a floor area of 20 m². He wants to enlarge the coup to make living conditions for the chickens more comfortable. He increases the width by a factor of 1,2 and he triples the length.
By what factor did the area of the chicken coup increase?
What is the perimeter of the new size chicken coup?
(Hint: let the width of the original coup be x and let the length be y)

QUESTION 2

2.1 Simplify the following:

2.1.1 $(4x)^{-3}/4x^{-2}$

2.1.2 $\sqrt[3]{(512x^{15})} + \sqrt[3]{(27x^{15})}$

2.1.3 $\sqrt{5}(\sqrt{5} + \sqrt{10}) - \sqrt{4}$ (Leave your answer in simplest surd form)

2.2 Solve for x:

2.2.1 $4.4^x = 1\ 024$

2.2.2 $7^x - 7^{x+1} = -42$

2.2.3 $2.3^{-x} = 54$

QUESTION 3

3.1 Consider the following patterns of circular pebbles:

5 14 27

3.1.1 How many pebbles are there in the next arrangement?

3.1.2 How many pebbles are there in the *n*th arrangement?

3.1.3 How many pebbles are there in the 16th arrangement?

3.1.4 Which arrangement has 779 pebbles?

3.2 Consider the sequence 1; *a*; 13; *b*; 33. The sequence has a constant second difference of 2.

3.2.1 Determine the values of *a* and *b*.

3.2.2 Calculate the *n*th term (general term) of the sequence.

QUESTION 4

4.1 Fazeela wants to start her own marketing firm and she decides to purchase a large printer to offer her clients printing services. The cost of the printer is R350 000.

4.1.1 Her partner Mohammed advises her that she needs to depreciate the printer at 18% per annum on a straight line basis. How long will it take her to write off the equipment?

4.1.2 Fazeela's accountant is of the opinion that she needs to depreciate the equipment at 25% per annum on the reducing balance and sell it after four years. Calculate the value of the printer after four years, and give your answer correct to the nearest cent.

4.2 Have a look at the two loans below. Calculate which loan will cost you less in the long run. Justify your answer by showing all your calculations.
Loan 1: 12% per annum, compounded half-yearly or
Loan 2: 10% per annum, compounded quarterly?

4.3 Danie inherits R78 000 and decides to invest the money he has inherited at an interest rate of 12% p.a. compounded monthly so that he can send both his children to university. Retha, his daughter, starts university exactly two years later, and he withdraws R65 000 to pay for her first year at university.
After he withdraws the money, the interest rate changes to 8% p.a. compounded quarterly, and the money stays invested for three more years at this rate. After the three years, the interest rate changes to 19% p.a. compounded bi-annually for two more years.
How much money will Danie have available to pay for his son's university tuition at the end of this investment period?

QUESTION 5

Given the functions $f(x) = -3/4\ (x + 3)^2 - 6$ and $g(x) = 3x + 4$:

5.1 Write down the coordinates of the turning point of f.

5.2 Calculate the roots of the equation $f(x) = 0$.

5.3 Write down the equation of the axis of symmetry of f.

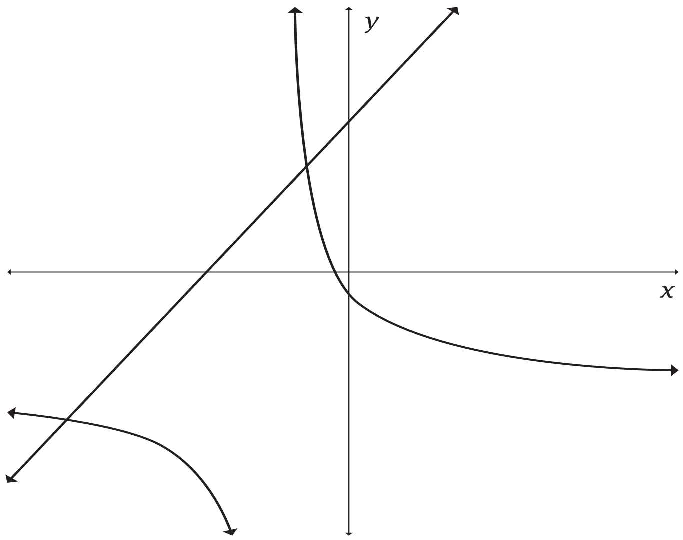
5.4 Sketch the graphs of y = f(x) and y = g(x) on the same system of axes.

5.5 Determine the equation of h(x) obtained by shifting f(x) three units to the right.

5.6 Determine the equation of k(x) obtained by shifting g(x) one unit up and two units to the right.

QUESTION 6

Sketched below are the graphs of $r(x) = 5/(x + 2) - 3$ and $s(x) = x + 4$.



6.1 Determine the coordinates of A and B.

6.2 Write down the equation of the horizontal asymptote of r(x).

6.3 Write down the domain of r(x).

6.4 Calculate the coordinates of the point where r(x) and s(x) intersect.

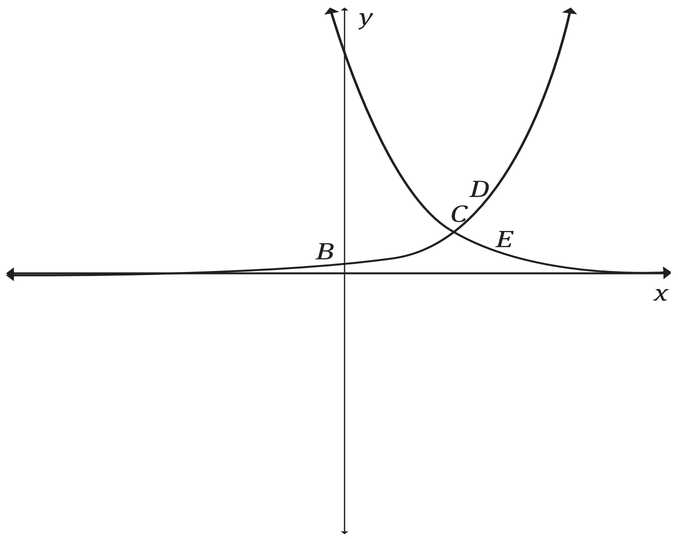
6.5 Write down the new equation if s(x) is reflected in the x-axis.

6.6 Write down the new equation if r(x) is reflected in the y-axis.

6.7 Determine the coordinates of H where GH is perpendicular to the x-axis.

QUESTION 7

Sketched below are the functions $m(x) = 3^{x-2}$ and $n(x) = 3^{1-x}$.
A and B are intercepts of the two graphs with the y-axis.
C is the point of intersection of the two graphs.
DEF is parallel to the y-axis with D and E on the two graphs.



7.1 Determine the length of AB.

7.2 Given that OF = 2 units, determine the average gradient between the points.

7.2.1 A and E

7.2.2 B and D

7.2.3 Hence determine which curve is steeper between $x = 0$ and $x = 2$.

7.3 Determine the coordinates of C.

Exam paper 2

MATHEMATICS PAPER 2

MARKS: 150 **TIME: 3 hours**

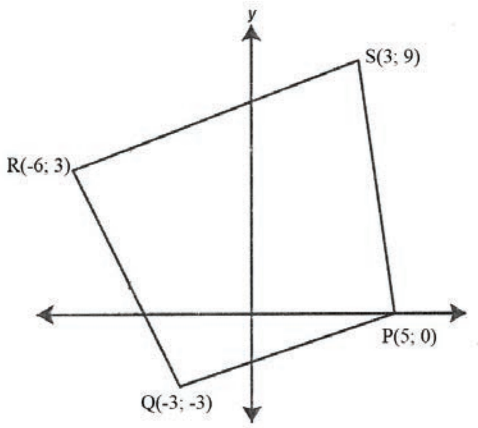
INSTRUCTIONS AND INFORMATION

Read the following instructions carefully before answering the question paper:

1 This question paper consists of 7 questions. Answer ALL questions.
2 Show ALL your calculations, diagrams, graphs, etcetera which you have used in answering the questions CLEARLY.
3 An approved scientific calculator (non-programmable and non-graphical) may be used unless stated otherwise.
4 Answers should be rounded off to TWO decimal places, unless stated otherwise.
5 Number your answers according to the numbering system used in this question paper.
6 Diagrams in the question paper are not necessarily drawn to scale.
7 It is in your own interest to write legibly (clearly) and to present your work neatly.

QUESTION 1

P(5; 0), Q(-3; -3), R(-6; 3) and S(3; 9) are the vertices of a quadrilateral in the Cartesian plane.



Determine:

1.1 The gradient of RQ.
1.2 The equation of RQ.
1.3 Is RQ parallel to SP? Show all your calculations.
1.4 The inclination of PQ.

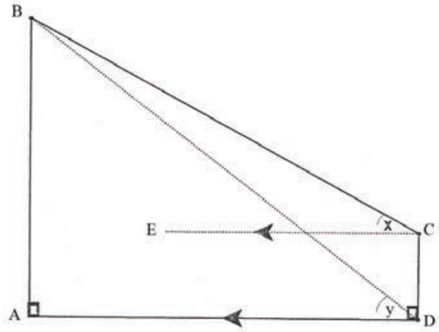
QUESTION 2

The vertices of a triangle in the Cartesian plane are A(-2; -2), B(4; 5) and C(4; 2). CD $\perp$ AB.

2.1 Determine the length of BC.
2.2 Determine the equation of CD.
2.3 What type of triangle is this? Give reasons for your answers.

QUESTION 3

An eagle hovering at point B can see two small creatures – a pigeon at point C and a mouse at point D. The angle of inclination from the mouse is y and from the pigeon to the eagle is x. The pigeon is 12 metres straight above the mouse and the eagle is straight above point A. The horizontal line CE is parallel to AD.



3.1 Determine the size of C^BD.

3.2 Determine the length of BD in terms of x and y.

3.3 Now determine the length of AB in terms of x and y.

3.4 if x = 38° and y = 43,21°, determine the actual length of AB, correct to two decimal places.

QUESTION 4

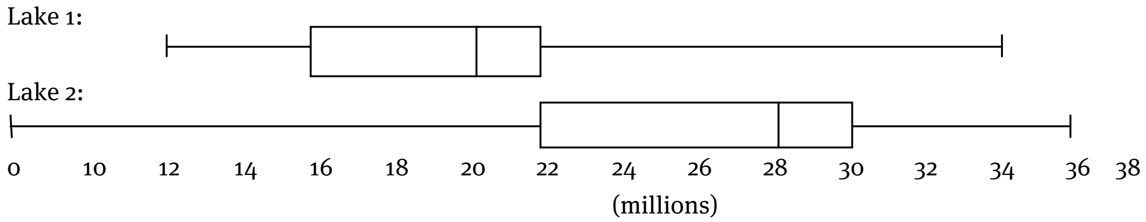
4.1 A hexagonal pizza has side lengths of 7 cm. Calculate the area of the pizza.

4.2 If 5 of the pizzas are placed on top of each other, and each subsequent pizza has side lengths of 0,5 mm less than that of the previous one, calculate the area of the pizza on top of the stack.

4.3 A packet of sweets has a hemisphere at the bottom with a cylinder on top. The diameter of the cylinder is 4 cm and the height is 10 cm. Determine the volume of the packet of sweets.

QUESTION 5

5.1 The box-and-whisker plots below depict the population density of brine shrimp in two salt-water lakes in the Americas, over a two year period.



5.1.1 Determine the five number summary for both populations.

5.1.2 Which lake had the biggest change in population numbers over the two years?

5.1.3 Write down the interquartile range for both data sets.

5.1.4 Explain the spread of the data for both lakes.

5.1.5 Which population is more consistent?

5.2 A survey was done to determine the age at which people move to retirement villages in the North West province. The results of the survey are tabled below:

Age	Frequency	Cumulative frequency
$40 \leq x < 50$	12	
$50 \leq x < 60$	27	

60 ≤ x < 70	58	
70 ≤ x < 80	215	
80 ≤ x < 90	143	
90 ≤ x < 100	42	

5.2.1 How many people were questioned in this survey?
5.2.2 Complete the table above.
5.2.3 Represent the information by using an ogive.
5.2.4 Determine the median age at which the people surveyed move to retirement villages.

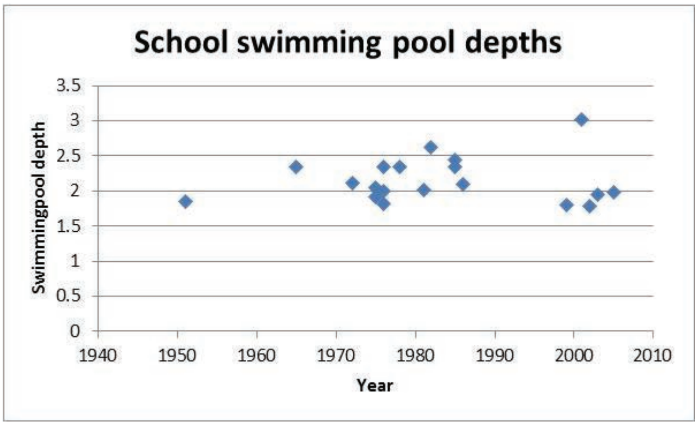
QUESTION 6

6.1 The depth at the deep end of 19 schools' swimming pools are given below:

2,12; 2,05; 2,63; 1,80; 1,85; 2,35; 2,44; 2,01; 3,02; 1,78; 1,95; 1,98; 2,09; 2,35; 1,82; 2,35; 1,99; 1,92; 2,35

6.1.1 Determine the mean, median and mode of the depths of the swimming pools.
6.1.2 Calculate the standard deviation of the depths using your calculator.
6.1.3 How many swimming pools fall within two standard deviation from the mean?

6.2 The swimming pools mentioned above were built at different time periods. A scatter diagram of the data that was produced is given below:



6.2.1 Sketch an approximate line of best fit on the data above.
6.2.2 What would you say is the depth of a school swimming pool built in 1979?
6.2.3 Use your approximate line of best fit to approximate the year in which a swimming pool with depth 2,25 m was built.

QUESTION 7

7.1 Evaluate tan 405° + cos 270°.sin 45°
7.2 Prove that $\sin x = \tan x\ (\cos^3 x + \cos x \sin^2 x)$
7.3 Determine the general solution of the equation $\sin 3x = -0{,}876$, correct to two decimal digits.

Answers to questions

Chapter 1

1   1.1 $3^{\frac{4}{3}} = 3^{\frac{4}{3}}$

1.2 $\sqrt[3]{3^3} = 3^{\frac{3}{3}} = 3^1 = 3$

1.3 $(3^{\frac{-3}{2}})^{\frac{2}{5}} = 3^{\frac{-6}{10}} = 3^{-\frac{3}{4}}$

1.4 $4^{\frac{3}{2}} - 36^{\frac{1}{2}} + 216^{\frac{1}{3}}$

$= (2^2)^{\frac{3}{2}} - (3^2 \times 2^2)^{\frac{1}{2}} + (3^3 \times 2^3)^{\frac{1}{3}}$

$= 2^{2 \times \frac{3}{2}} - (3^{2 \times \frac{1}{2}} \times 2^{2 \times \frac{1}{2}}) + (3^{3 \times \frac{1}{3}} \times 2^{3 \times \frac{1}{3}})$

$= 2^{\frac{6}{2}} - (3^{\frac{3}{2}} \times 2^{\frac{2}{2}}) + (3^{\frac{3}{3}} \times 2^{\frac{3}{3}})$

$= 2^3 - (3^1 \times 2^1) + (3^1 \times 2^1)$ $\quad = 2^3 - 6 + 6$ $\quad = 8$

1.5 $(0{,}0625)^{-\frac{3}{4}} . (0{,}0125)$

$= (0{,}5^4)^{-\frac{3}{4}} . (0{,}5^3)$ $\quad = (0{,}5^{-\frac{12}{4}}) . (0{,}5^3)$

$= 0{,}5^{-3} . 0{,}5^3$ $\quad = 0{,}5^0$ $\quad = 1$

1.6 $\sqrt[6]{(64n^{12})^2} = \sqrt[6]{(2^6n^{12})^2} = \sqrt[6]{(2^{12}n^{24})} = 2^2n^4 = 4n^4$

1.7 $(49m^7n^9)^{\frac{6}{4}} = (7^2m^7n^9)^{\frac{6}{4}} = 7^{\frac{12}{4}}m^{\frac{42}{4}}n^{\frac{54}{4}} = 7^3m^{10{,}5}n^{13{,}5} = 343\, m^{10{,}5}n^{13{,}5}$

1.8 $(81x^3y^7)^{-\frac{2}{3}} .3(x^{-4}y^{-3})^{\frac{2}{-3}} .$

$= (3^4x^3y^7)^{-\frac{2}{3}} . 3(x^{-4}y^{-3})^{-\frac{2}{3}} = (3^{-\frac{8}{3}}x^{-\frac{6}{3}}y^{-\frac{14}{3}}) . 3(\frac{8}{x^3}y^{\frac{6}{3}})$

$= 3^{-\frac{8}{3}}x^{-2}y^{-\frac{14}{3}} \times 3x^{\frac{8}{3}}y^2 = \dfrac{3x^{\frac{2}{3}}}{3^{\frac{8}{3}}y^{\frac{8}{3}}}$

1.9 $[\dfrac{\sqrt{169x^3y^4}}{(7x^{-3})^{-4}}]^{-1} = [\dfrac{13\sqrt{x^3}y^2}{(7^{-4}x^{12})}]^{-1} = \dfrac{7^{-4}x^{12}}{13(x^3)^{\frac{1}{2}}y^2} = \dfrac{\frac{1}{7^4}x^{12}}{13(x^{\frac{3}{2}})y^2} = \dfrac{x^8}{\frac{1}{31213\, y^2}}$

1.10 $(4^{2n+3})^{\frac{1}{7}} . \dfrac{(7^{2n-3})^{\frac{1}{7}}}{(12^{2n-3})^{\frac{1}{7}}} . (5^{2n+3})^{\frac{1}{7}} = (2^2)^{\frac{2}{7n}+\frac{3}{7}} . \dfrac{7^{\frac{2}{7}n-\frac{3}{7}}}{(3.2^2)^{\frac{2}{7}n-\frac{3}{7}}} . 5^{\frac{2}{7}n+\frac{3}{7}} =$

$\left(2^{\frac{4}{7n}+\frac{6}{7}}\right)\left(\dfrac{7^{\frac{2}{7n}-\frac{3}{7}}}{3^{\frac{2}{7n}-\frac{3}{7}}.2^{\frac{4}{7n}-\frac{6}{7}}}\right).(5^{\frac{2}{7n}+\frac{3}{7}})$

$= 2^{\frac{4}{7n}+\frac{6}{7}} . 7^{\frac{2}{7n}-\frac{3}{7}} . 5^{\frac{2}{7n}+\frac{3}{7}} . 3^{\frac{-2}{7n}-\frac{3}{7}} . 2^{\frac{-4}{7n}+\frac{6}{7}} = 2^{\frac{12}{7}} . 7^{\frac{2}{7n}-\frac{3}{7}} . 5^{\frac{2}{7n}+\frac{3}{7}} . 3^{\frac{-2}{7n}-\frac{3}{7}}$

2   2.1 $y^{\frac{2}{3}} = 3^2$

$(y^{\frac{2}{3}})^{\frac{3}{2}} = (3^2)^{\frac{3}{2}}$ $\quad y = 3^3$ $\quad y = 27$

2.2 $k^{\frac{1}{5}} = 3$ $(k^{\frac{1}{5}})^{\frac{5}{1}} = 3^5$ $k = 243$

2.3 $m^{\frac{-4}{3}} = 0{,}0625$

$(m^{\frac{-4}{3}})^{\frac{-3}{4}} = (0{,}5^4)^{\frac{-3}{4}}$

$m = 0{,}5^{\frac{-12}{4}}$ $= 0{,}5^{-3}$ $= \frac{1}{0{,}5^3}$

2.4 $2^{y+3} + 2^y = 9$

$2^y(2^3 + 1) = 9$

$2^y(9) = 9$

$2^y = 1$

$2^y = 2^\circ$

$y = 0$

2.5 $7^{-k} - 7^{-k-2} = 48$

$7^{-k}(1 - 7^{-2}) = 48$

$7^{-k}\left(1 - \frac{1}{49}\right) = 48$

$7^{-k}\left(\frac{48}{49}\right) = 48$

$7^{-k} = \frac{48}{\frac{48}{49}}$

$7^{-k} = 48 \times \frac{49}{48}$

$7^{-k} = 49$

$7^{-k} = 7^2$

$-k = 2$

$k = -2$

2.6 $2^{\frac{x}{2}} + 2^{\frac{x}{2}+1} = 24$

$2^{\frac{x}{2}} + (1 + 2^1) = 24$

$2^{\frac{x}{2}}(3) = 24$

$2^{\frac{x}{2}} = 8$

$2^{\frac{x}{2}} = 2^3$

$\frac{x}{2} = 3$

$x = 6$

2.7 $(7^x + 14)(2^x - 0{,}0875) = 0$

$7^x = -14$ or $2^x = 0{,}0875$

$7^x = -14$ $\qquad$ $2^x = \frac{875}{10000}$

no solution $\qquad$ $= \frac{5^3.7}{5^4.4^2}$ $\qquad$ $= \frac{7}{5\,.4^2}$

$= \frac{7}{80}$

2.8 $3^{2x} - 2\,.\,3^x = 3$

$3^{2x} - 2\,.3^x - 3 = 0$

$(3^x - 3)(3^x + 1) = 0$

$3^x = 3$ or $3^x = -1$

$3^x = 3^1$ or $3^x = -3^{\circ}$

$x = 1$ $\qquad$ $x = 0$

2.9 $16^x + 8\,.4^x = 48$

$2^{4x} + 2^3\,.\,2^{2x} = 48$ let $2^{2x} = k$

$k^2 + 2^3\,.\,k = 48$

$k^2 + 8k - 48 = 0$

$(k - 4)(k + 12) = 0$

$k = 4$ or $k = -12$

$2^{2x} = 4$ or $2^{2x} = -12$

$2x = 2$ $\qquad$ no solution

$x = 1$

2.10 $7^{-x+2} + 7^{2+x} = 392$

$7^{-x}\,.\,7^3 + 7^2\,.\,7^x = 392$

$\frac{1}{7^x}\,.\,7^3 + 7^2\,.\,7^x = 392$

Let $\qquad 7^x = k$

$\frac{1}{k}\,.\,7^3 + 7^2 k = 392$

$49k^2 + 343 - 392k = 0$

$k^2 - 8k + 7 = 0$

$(k - 7)(k - 1) = 0$

$k = 7$ or $k = 1$

$7^x = 7$ or $7^x = 1$

$x = 1$ $\qquad$ $7^x = 7^{\circ}$ $\qquad$ $x = 0$

3 3.1 $\sqrt[3]{5} + 9\sqrt[3]{5} - 4\sqrt[3]{5} = 6\sqrt[3]{5}$

3.2 $12\sqrt{9} - 3\sqrt{45} + 6\sqrt{72}$

$= 12\sqrt{3^2} - 3\sqrt{5.3^2} + 6\sqrt{3^2 2^3}$

$= (12\,.3) - \left(3\,.3\sqrt{5}\right) + (6\,.\,3\,.\,2\,.\sqrt{2})$

$= 36 - 9\sqrt{5} + 36\sqrt{2}$

3.3 $\frac{(\sqrt{50} - \sqrt{72})}{\sqrt{98}}$

$= \frac{\sqrt{5^2.2} - \sqrt{3^2\,.\,2^3}}{\sqrt{7^2.2}}$

$= \frac{5\sqrt{2} - 3\,.2\sqrt{2}}{7\sqrt{2}}$

$= \frac{-\sqrt{2}}{7\sqrt{2}}$

$= -\frac{1}{7}$

3.4 $\sqrt[5]{\sqrt{a^{100}}} + \sqrt[3]{\sqrt{a^{15}}} - \sqrt[7]{\sqrt{a^8}}$

$= \sqrt[10]{a^{100}} + \sqrt[6]{a^{15}} - \sqrt[14]{a^8}$

$= a^{10} + a^{\frac{15}{6}} - 9^{\frac{8}{14}}$

3.5 $\sqrt{18} - \sqrt{80} + \sqrt{98}$

$= \sqrt{3^2.2} - \sqrt{2^4.5} + \sqrt{7^2.2}$

$= 3\sqrt{2} - 4\sqrt{5} + 7\sqrt{2}$

$= 10\sqrt{2} - 4\sqrt{5}$

3.6 $\frac{\sqrt{\sqrt{64} - \sqrt{48}}}{\sqrt{50}}$

$= \frac{\sqrt{8 - \sqrt{2^4\,.3}}}{\sqrt{5^2\,.2}}$

$= \frac{\sqrt{2^3 - 4\sqrt{3}}}{5\sqrt{2}}$

$= (2^3 - 2^2\left(3^{\frac{1}{2}}\right))^{\frac{1}{2}}$

$= 2^{\frac{3}{2}} - 2^1\,(3^{\frac{1}{4}})$

$= 2^1\,.\,2^{\frac{1}{2}} - 2^1(3^{\frac{1}{4}})$

$= \frac{2(\sqrt{2} - \sqrt[4]{3})}{5\sqrt{2}}$

4 4.1 $3\sqrt{5} \times 2\sqrt{2} = 6\sqrt{10}$

4.2 $-4\sqrt{3} \times (-5\sqrt{2}) = 20\sqrt{6}$

4.3 $4\sqrt{5}\,(-2\sqrt{2} + 3) = -8\sqrt{10} + 12\sqrt{5}$

4.4 $2\sqrt{2}\,(10\sqrt{3} - 8\sqrt{2}) = 20\sqrt{6} - 16\sqrt{4} = 20\sqrt{6} - 32$

4.5 $\frac{\sqrt[4]{48x^4}}{y^{24}} = (\frac{48x^4}{y^{24}})^{\frac{1}{4}} = \frac{48^{\frac{1}{4}}x}{y^6} = \frac{(2^4.3)^{\frac{1}{4}}x}{y^6} = \frac{2.3^{\frac{1}{4}}x}{y^6}$

4.6 $\sqrt[3]{\frac{125x^{12}}{y^6}} = \frac{5x^4}{y^2}$

4.7 $\sqrt[4]{\frac{3}{243}} = \sqrt[4]{\frac{3}{3^5}} = \sqrt[4]{\frac{1}{3^4}} = \frac{1}{3}$

4.8 $\frac{(k-2)}{k^2}$ if $k = 1 + \sqrt{3}$

$= \frac{(1+\sqrt{3}-2)}{(1+\sqrt{3})^2} = \frac{\sqrt{3}-1}{(1+2\sqrt{3}+3)} = \frac{\sqrt{3}-1}{2\sqrt{3}+4}$

5 5.1 $\sqrt{x-1} = 2$

$x - 1 = 4$

$x = 5$

5.2 $\sqrt{x+3} = 5$

$x + 3 = 25$

$x = 22$

5.3 $\sqrt{x-1} = 4$

$x - 1 = 16$

$x = 17$

5.4 $x - \sqrt{-8x-16} = 5$

$-\sqrt{-8x-16} = 5 - x$

$-(-8x - 16) = (5 - x)^2$

$8x + 16 = x^2 - 2x + 25$

$0 = x^2 - 10x + 9$

$0 = (x - 9)(x - 1)$

$x = 9$ or $x = 1$

5.5 $\sqrt{x+7} = -7$

$x + 7 = -7^2$

$x = -49 - 7$

$x = -56$

5.6 $\sqrt{11} + \sqrt{y} = \sqrt{y-2}$

$11 + y = y - 2$

$+y - y = -13$

no solution

Chapter 2

1 1.1 $(x-2)(x+7) = 0$
$x - 2 = 0$ or $x + 7 = 0$
$x = 2$ or $x = -7$

1.2 $(2y-3)(y+5) = 0$
$2y - 3 = 0$ or $y = 5 = 0$
$2y = 3$ $y = -5$
$y = \frac{3}{2}$

1.3 $x^2 + 21x + 10 = 0$
$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
$= \frac{-21 \pm \sqrt{21^2-4(1)(10)}}{2(1)}$
$= \frac{-21 \pm 20{,}02}{2}$
$x = -0.49$ or $x = -20{,}51$

1.4 $3k(k+4) = 0$
$3k = 0$ or $k + 4 = 0$
$k = 0$ $k = -4$

1.5 $9x^2 - 5x = 0$
$x(9x - 5) = 0$
$x = 0$ or $9x - 5 = 0$
$9x = 5$
$x = \frac{5}{9}$

1.6 $3k(1-k) = 5(k+1) = 0$
$3k - 3k^2 + 5k + 6 = 0$
$-3k^2 + 8k + 6 = 0$
$3k^2 - 8k - 6 = 0$
$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
$= \frac{-(-8) \pm \sqrt{(-8)^2-4(3)(-6)}}{2(3)}$
$= \frac{8 \pm 11.66}{6}$

$x = 3.28$ or $x = -0.61$

1.7 $(3p-2)(p+1)+2=0$
$3p^2+p-2+2=0$
$3p^2+p=0$
$p(3p+1)=0$
$p=0$ or $3p=-1$
$p=\frac{-1}{3}$

1.8 $b(b+5)=6$
$b^2+5b-6=0$
$(b-1)(b+6)=0$
$b=1$ or $b=-6$

1.9 $(x-2)(x+2)=6(3x+5)$
$x^2-4=18x+30$
$x^2-18x-34=0$
$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
$\frac{-(-18)\pm\sqrt{(-18)^2-4(1)(-34)}}{2(1)}$
$\frac{18\pm21.45}{2}$
$x=19{,}72$ or $x=-1{,}725$

1.10 $4(x-1)(x+1)=\ 3(2-x)+5$
$4(x^2-1)=6-3x+5$
$4x^2+3x-15=0$
$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
$=\frac{-3\pm\sqrt{15{,}78}}{8}$
$x=1{,}60$ or $x=-2{,}35$

2 2.1 $\frac{5}{x-1}=\frac{x}{x+1}$
$5(x+1)=\ x(x-1)$
$5x+5=x^2-x$
$0=x^2-6x-5$
$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
$=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(-5)}}{2(1)}$
$\frac{6\pm7{,}48}{2}$
$x=6{,}74$ or $x=-0{,}74$

2.2 $\frac{3}{2x-6}+\frac{x}{x-3}=0$
$3(x-3)+x(2x-6)=0$
$3x-9+2x^2-6x=0$
$2x^2-3x-9=0$
$(2x+3)(x-3)=0$
$2x+3=0$ or $x-3=0$
$x=\frac{-3}{2}$ or $x=3$

2.3 $\frac{(x+2)}{(x-3)}=7+\ \frac{2}{(x-3)}$

$x+2=7(x-3)+\ 2$

$x + 2 = 7x - 21 + 2$

$21 = 6x$

$\frac{21}{6} = x$

3 3.1 $\sqrt{(6x + 5)} = x$

$(\sqrt{(6x + 5)})^2 = x^2$

$6x + 5 = x^2$

$0 = x^2 - 6x - 5$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-5)}}{2}$

$\frac{6 \pm 7{,}48}{2}$

$x = 6{,}74$ or $x = -0{,}74$

3.2 $x = \pm\sqrt{27}$

3.3 $\sqrt{2x + 3} - x = 0$

$\sqrt{2x + 3} = x$

$(\sqrt{2x + 3})^2 = x^2$

$2x + 3 = x^2$

$0 = x^2 - 2x - 3$

$0 = (x - 3)(x + 1)$

$x = 3$ or $x = -1$

3.4 $\sqrt{x + 9} + x + 3 = 0$

$(\sqrt{x + 9})^2 = (-x - 3)^2$

$x + 9 = x^2 - 6x + 9$

$0 = x^2 - 7x$

$0 = x\,(x - 7)$

$x = 0$ or $x = 7$

3.5 $2 = \sqrt{(x^2 - 27)}$

$2^2 = \sqrt{x^2 - 27})^2$

$4 = x^2 - 27$

$0 = x^2 - 31$

$31 = x^2$

$\pm\sqrt{31} = x$

3.6 $\sqrt{(x-1)} = \sqrt{(4x-2)}$

$\left(\sqrt{x-1}\right)^2 = \left(\sqrt{4x-2}\right)^2$

$x - 1 = 4x - 2$

$1 = 3x$

$\frac{1}{3} = x$

4 4.1 $x^2 = 81$

$\sqrt{x^2} = \sqrt{81}$

$x = \pm 9$

4.2 $x^2 = 27$

$\sqrt{x^2} = \sqrt{27}$

$x = \pm\sqrt{27}$

$= \pm 5{,}20$

4.3 $x^2 - 16 = 0$

$\sqrt{x^2} = \sqrt{16}$

$x = \pm 4$

4.4 $-x^2 + 49 = 0$

$\sqrt{49} = \sqrt{x^2}$

$\pm 7 = x$

4.5 $(x+4)^2 = 48$

$\sqrt{(x+4)^2} = \sqrt{48}$

$x + 4 = \pm 6{,}93$

$x = \pm 2{,}93$

4.6 $5(x+5)^2 = 125$

$\sqrt{(x+5)^2} = \sqrt{25}$

$x + 5 = \pm 5$

$x = -10$ or $x = 0$

4.7 $3\,(x+4)^2 - 12 = 0$

$3\,(x+4)^2 = 12$

$\sqrt{(x+4)^2} = \sqrt{4}$

$x + 4 = \pm 2$

$x = -2$ or $x = -6$

4.8 $x^2 = (2x - 3)^2$

$\sqrt{x^2} = \sqrt{(2x-3)^2}$

$x = 2x - 3$

$0 = x - 3$

$3 = x$

4.9 $3\,(x - 2)^2 - 16 = 2$

$3\,(x - 2)^2 = 18$

$\sqrt{(x-2)^2} = \sqrt{6}$

$x - 2 = \pm 2{,}45$

$x = 4{,}45$ or $x = -0{,}45$

4.10 $-x = (\frac{1}{2}x + 2)^2 - 6$

$-x + 6 = \frac{1}{4}x^2 + 3x + 4$

$0 = \frac{5}{4}x^2 + 3x + 10$

5 5.1 $(x^2 - 3x)^2 - 2(x^2 - 3x) - 5 = 0$

Let $(x^2 - 3x) = k$

$(k)^2 - 2(k) - 5 = 0$

$k^2 - 2k - 5 = 0$

$k = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{\left(-(-2) \pm \sqrt{(-2)^2 - 4(1)(-5)}\right)}{2}$

$\frac{2 \pm 4{,}90}{2}$

$k = 3{,}45$ or $k = -1{,}45$

$x^2 - 3x = 3{,}45$ or $x^2 - 3x = -1{,}45$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ or $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(3{,}45)}}{2}$ or $= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-1{,}45)}}{2}$

$= \frac{3 \pm \sqrt{-4,8}}{2}$ or $= \frac{3 \pm 14,8}{2}$

no solution $x = 8,9$ or

$x = -5,9$

5.2 $(x^2 - 2x)^2 = 14\,(x^2 - x) + 15$

let $x^2 - 2x = k$

$(k)^2 = 14(k) + 15$

$k^2 - 14k - 15 = 0$

$(k - 15)(k + 1) = 0$

$k = 15$ or $k = -1$

$x^2 - 2x = 15$ or $x^2 - 2x = -1$

$x^2 - 2x - 15 = 0$ $x^2 - 2x + 1 = 0$

$(x - 5)(x + 3) = 0$ $(x - 1)(x - 1) = 0$

$x = 5$ or $x = -3$ $x = 1$

5.3 $\sqrt{x - 2} + 4 = \frac{5}{\sqrt{x-2}}$

Let $\sqrt{x - 2} = k$

$k + 4 = \frac{5}{k}$

$k^2 + 4k = 5$

$k^2 + 4k - 5 = 0$

$(k + 1)(k - 1) = 0$

$k = -5$ or $k = 1$

$\sqrt{x - 2} = -5$ or $\sqrt{x - 2} = 1$

$x - 2 = (-5)^2$ $x - 2 = 1^2$

$x = 27$ $x = 3$

5.4 $\left(\frac{3}{x} + x\right)^2 + \left(\frac{3}{x} + x\right) = 19$

let $\frac{3}{x} + x = k$

$k^2 + k - 19 = 0$

$k = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-1 \pm \sqrt{1^2 - (4)(1)(-19)}}{2}$

$= \frac{-1 \pm 8{,}77}{2}$

$k = 3{,}89$ or $k = -4{,}89$

$\frac{3}{x} + x = 3{,}89$ or $\frac{3}{x} + x = -4{,}89$

$3 + x^2 - 3{,}89x$ $\quad$ $3 + x^2 + 4{,}89x = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\quad$ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-(-3{,}89) \pm \sqrt{(-3{,}89)^2 - 4(3)}}{2}$ $\quad$ $= \frac{-(4{,}89) \pm \sqrt{(4{,}89)^2 - 4(3)}}{2}$

$= \frac{3{,}89 \pm 1{,}77}{2}$ $\quad$ $= \frac{4{,}89 \pm 11{,}91}{2}$

$x = 2{,}83$ or $x = 1{,}06$ $\quad$ $x = 8{,}40$ or $x = -3{,}51$

5.5 $5(2x^2 + x - 1) = 20(2x^2 + x - 1) + 8$

let $2x^2 + x - 1 = k$

$5k = 20k + 8$

$0 = 15k + 8$

$\frac{-8}{15} = k$

but $k = 2x^2 + x - 1$

$\frac{-8}{15} = 2x^2 + x - 1$

$0 = 2x^2 + x - \frac{7}{15}$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-1 \pm \sqrt{(-1)^2 - (4)(2)(-\frac{7}{15})}}{2(2)}$

$= \frac{-1 \pm 2{,}18}{4}$

$x = 0{,}295$ or $x = -0{,}795$

5.6 $2(x + 3)^2 - 3(x + 3) - 4 = 2$

let $x + 3 = k$

$2k^2 - 3k - 6 = 0$

$k = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-6)}}{2(2)}$

$= \frac{3 \pm 7{,}55}{4}$

$k = 2,64$ or $k = -1,14$

but $k = x + 3$

$x + 3 = 2,64$ or $x + 3 = -1,14$

$x = -0,36$ $x = -4,14$

5.7 $\frac{24}{3(x-2)} = \frac{7}{9(x+6)} - 3$

$24\big(9(x+6)\big) = 7\big(3(x-2)\big) - 3\big(3(x-2)\big)\big(9(x+6)\big)$

$216x + 1296 = 21x - 42 - 3(27x^2 + 162x - 54x - 324)$

$216x + 1296 = 21x - 42 - 81x^2 - 486x + 162x + 972$

$81x^2 + 519x + 366 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\frac{-519 \pm \sqrt{519^2 - 4(81)(366)}}{2(81)}$

$\frac{-519 \pm 388,30}{162}$

$x = 1,42$ or $x = -3,38$

6 6.1 $p^2 - p - 12 = 0$

$(p-4)(p+3) = 0$

$p = 4$ or $p = -3$

6.2 $x^2 + 3x = 4$ $x^2 + 3x = -3$

$x^2 + 3x - 4 = 0$ $x^2 + 3x + 3 = 0$

$(x+4)(x-1) = 0$ $x = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = -4$ or $x = 1$ $= \frac{3 \pm \sqrt{3^2 - 4(1)(3)}}{2}$

no solution

7 $(a+3)(b+4) = 0$ $a = -7$

$(-4)(b+4) = 0$

$-4b - 16 = 0$

$-4b = 16$

$b = -4$

8 8.1 $(y-2)((-7)^2 + 25(-7) - 6) = 0$

$(y-2)(-132)=0$

$-132y+264=0$

$y=2$

8.2 $(y-2)(12^2+25(12)-6)=0$

$(y-2)(438)=0$

$438y-876=0$

$y=2$

9 9.1 $x^2+\ 2x+4=0$

$x^2+\ 2x=\ -4$

$x^2+\ 2x+(\frac{b}{2})^2=\ -4+\ \left(\frac{b}{2}\right)^2$

$x^2+\ 2x+1=\ -4+1$

$(x+1)^2=\ -3$

$\sqrt{(x+1)^2}=\ \pm\sqrt{-3}$

$x+1=\ \pm\sqrt{-3}$

$x=\ -1\ \pm\sqrt{-3}$

$x=0{,}73$ or $x=\ -2{,}73$

9.2 $x^2-\ 5x+15=0$

$x^2-\ 5x=\ -15$

$x^2-\ 5x+\left(\frac{b}{2}\right)^2=\ -15+(\frac{b}{2})^2$

$x^2-\ 5x+\left(\frac{5}{2}\right)^2=\ -15+\ \left(\frac{5}{2}\right)^2$

$\sqrt{\left(x-\ \frac{5}{2}\right)^2}=\ \pm\sqrt{-8{,}75}$

$x-\ \frac{5}{2}=\ \pm\sqrt{-8{,}75}$

$x=\ \frac{5}{2}\ \pm\sqrt{-8{,}75}$

no solution

9.3 $x^2-\ x+20=0$

$x^2-\ x+(\frac{-1}{2})^2=\ +20+(\frac{-1}{2})^2$

$\sqrt{(x-\left(\frac{+1}{2}\right))^2}=\ \pm\sqrt{20{,}25}$

$x - \frac{1}{2} = \pm\sqrt{20{,}25}$

$x = \frac{1}{2} \pm 4{,}5$

$x = 5$ or $x = -4$

9.4 $x^2 + 6x = -5$

$x^2 + 6x + (\frac{b}{2})^2 = -5 + (\frac{b}{2})^2$

$x^2 + 6x + \left(\frac{6}{2}\right)^2 = -5 + 9$

$\sqrt{(x+3)^2} = \pm\sqrt{4}$

$x + 3 = \pm 4$

$x = -3 \pm 4$

$x = 1$ or $x = -7$

9.5 $13x = 10 + x^2$

$-10 = -13x + x^2$

$-10 + (\frac{-13}{2})^2 = (\frac{-13}{2})^2 - 13x + x^2$

$\sqrt{32{,}25} = \sqrt{(x - \frac{13}{2})^2}$

$\pm 5{,}68 = x - \frac{13}{2}$

$\frac{13}{2} \pm 5{,}68 = x$

$x = 12{,}18$ or $x = 0{,}82$

9.6 $x^2 - 6x = 3$

$x^2 - 6x + (\frac{-6}{2})^2 = 3 + (\frac{-6}{2})^2$

$\sqrt{(x-3)^2} = \pm\sqrt{12}$

$x - 3 = \pm\sqrt{12}$

$x = 3 \pm 3{,}46$

$x = 6{,}46$ or $x = 0{,}46$

Chapter 3

1 1.1 i 2

ii 49; 64; 81; 100; 121

		iii	$T_n = n^2 + 4n + 4$
		iv	$T_{12} = 12^2 + 4(12) + 4 = 196$
			$T_{30} = 30^2 + 4(30) + 4 = 1\ 024$
	2.2	i	6
		ii	84; 119; 160; 207; 260
		iii	$T_n = 3n^2 + 2n - 1$
		iv	$T_{12} = 3(12)^2 + 2(12) - 1 = 455$
			$T_{30} = 3(30)^2 + 2(30) -1 = 2\ 759$
	2.3	i	2
		ii	73; 96; 121; 148; 177
		iii	$T_n = n^2 + 12n - 12$
		iv	$T_{12} = 12^2 + 12(12) -12 = 276$
			$T_{30} = 30^2 + 12(30) -12 = 1\ 248$
	2.4	i	10
		ii	92; 141; 200; 269; 348
		iii	$T_n = 5n^2 - 6n - 3$
		iv	$T_{12} = 5(12)^2 -6(12) -3 = 645$
			$T_{30} = 5(30)^2 -6(30) -3 = 4\ 317$
	2.5	i	34
		ii	321; 485; 683; 915; 1 181
		iii	$T_n = 17n^2 - 23n + 11$
		iv	$T_{12} = 17(12)^2 - 23(12) + 11 = 2\ 183$
			$T_{30} = 17(30)^2 - 23(30) + 11 = 14\ 621$
2	2.1	$T_1 = 1$	
		$T_2 = 3$	
		$T_3 = 7$	

$T_4 = 13$

$T_5 = 21$

$T_6 = 31$

$T_7 = 43$

2.2 $T_n = n^2 - n + 1$

2.3 $133 = n^2 - n + 1$

$\therefore n^2 - n - 132 = 0$

$\therefore (n - 12)(n + 11) = 0$

$\therefore n = 12$ or $n = -11$ (not possible)

$\therefore 133$ is T_{12}

$1\,407 = n^2 - n + 1$

$\therefore n^2 - n - 1\,406 = 0$

$\therefore (n - 38)(n + 37) = 0$

$\therefore n = 38$ or $n = -37$ (not possible)

$\therefore 1\,407$ is T_{38}

Chapter 4

1 1.1 $\tan\theta = m$

$\tan\theta = 15$

$\theta = 86{,}19^0$

1.2 $\tan \theta = m$

$\tan\theta = 12/6$

$\theta = 63{,}43^0$

1.3 $\tan\theta = m$

$\tan\theta = -3/4$

$\theta = -36{,}87^0$

1.4 $\tan\theta = m$

$\tan\theta = -3$

$\theta = -71{,}57^0$

1.5 $\tan\theta = m$

$\tan\theta = -0{,}125$

$\theta = -7,13^0$

1.6 $m_{QR} = y_2 - y_1 / x_2 - x_1$

$= -7-2/-4-(-6)$

$= -9/2$

$\tan\theta = -9/2$

$\theta = -77,47^0$

1.7 $m_{QR} = y_2 - y_1 / x_2 - x_1$

$= -3-13/8-5/7$

$= -16/(51/7)$

$\tan\theta \quad = -16/(51/7)$

$\theta = -65,52^0$

1.8 $m_{QR} \quad = y_2 - y_1 / x_2 - x_1$

$= -1-(-4)/-3-1$

$= 3/-4$

$\tan\theta \quad = -3/4$

$\theta = -36,87^0$

1.9 $m_{QR} \quad = y_2 - y_1 / x_2 - x_1$

$= -4-(-1)/-2-(-6)$

$= -3/4$

$\tan\theta \quad = -3/4$

$\theta = -36,87^0$

1.10 $m_{QR} \quad = y_2 - y_1 / x_2 - x_1$

$= --1-7/5-(-12)$

$= -8/17$

$\tan\theta \quad = -8/17$

$\theta = --25.20^0$

2 2.1 $\tan\theta = m$

$\tan 72^0 = m$

$3.08 = m$

2.2 $\tan\theta = m$

$\tan 250^0 = m$

$2,75 = m$

2.3 $\tan\theta = m$

$\tan 13^0 = m$

$0,23 = m$

2.4 $\tan\theta = m$

$\tan -126^0 = m$

$1.38 = m$

2.5 $\tan\theta = m$

$\tan -50,8^0 = m$

$-1.23 = m$

2.6 $\tan\theta = m$

$\tan 185,15^0 = m$

$1,38 = m$

3 3.1 $m_{AB} = y_2 - y_1 / x_2 - x_1$ $m_{CD} = y_2 - y_1 / x_2 - x_1$

$= 5-8/-1-7$ $= -2-3/5-16$

$= 3/-8$ $= -5/-11$

Neither.

3.2 $m_{AB} = y_2 - y_1 / x_2 - x_1$ $m_{CD} = y_2 - y_1 / x_2 - x_1$

$= -5-2/-5-7$ $= -1-6/-5-7$

$= -7/-12$ $= -5/-12$

$\therefore m_{AB} = m_{CD}$

Parallel.

3.3 $m_{AB} = y_2 - y_1 / x_2 - x_1$ $m_{CD} = y_2 - y_1 / x_2 - x_1$

$= 6-(-7)/-9-(-5)$ $= --3-7/9-4$

$= 1/-4$ $= -10/5 = -2$

Neither.

3.4 $m_{AB} = y_2 - y_1 / x_2 - x_1$ $m_{CD} = y_2 - y_1 / x_2 - x_1$

$= 5-2/2-3$ $= 3-3/-9-1$

$= -3$ $= 0$

Neither.

3.5 $m_{AB} = y_2 - y_1 / x_2 - x_1$ $m_{CD} = y_2 - y_1 / x_2 - x_1$

$= 1-4/9-8$ $= 3-5/-4-2$

$= -3$ $= 1/3$

$m_{AB} \times m_{CD}$

$= -3 \times 1/3$

$= -1$

AB is perpendicular to CD.

4 4.1 $y = mx + c$

$y = 5x - 1{,}7$

4.2 $y = mx + c$

$y = -2/5x + 9$

4.3 $y - y_1 = m(x - x_1)$

$y - 5 = 7/3\ (x-(-13))$

$y = 7/3x + 106/3$

4.4 $y - y_1 = m(x - x_1)$

$y-(-1) = -5(x-(-2))$

$y= -5x -11$

4.5 $m = y_2-y_1/x_2-x_1$

$=5-(-1)/-4-(-1)$

$=-2$

$y- y_1 = m(x-x_1)$

$y-5= -2(x-(-4))$

$y= -2x -3$

4.6 $m = y_2-y_1/x_2-x_1$

$=-12-9/13-(-5)$

$=-21/18$

$y- y_1 = m(x-x_1)$

$y-(-12)= -21/18(x-13)$

$y= -21/18x + 19/6$

4.7 $5y= x-15$

$y=1/5x -3$

$m= 1/5$

Passes through (-1; 4).

$y- y_1 = m(x-x_1)$

$y-4= 1/5(x- (-1))$

$y= 1/5 + 21/5$

4.8 $2y= 6x -7$

$y= 2x -7/2$

KT is perpendicular to BS so:

$m_{KT} \times m_{BS} = -1$

$m_{KT} \times 2 = -1$

$m_{KT} = -1/2$

Passes through (3/4; 0,5).

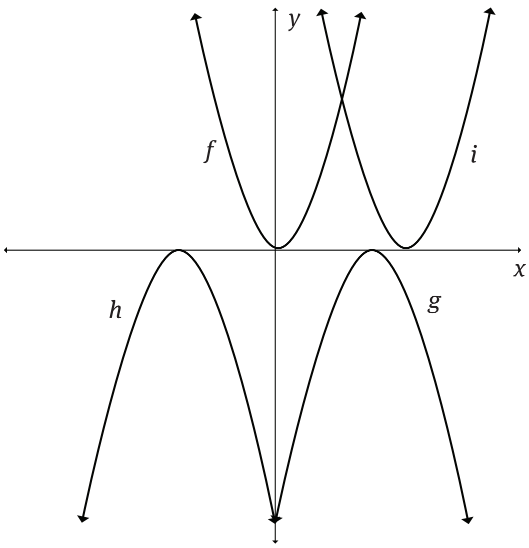
$y - y_1 = m(x - x_1)$

$y - 0{,}5 = -1/2(x - 4/3)$

$y = -1/2x + 13/6$

Chapter 5

1 1.1



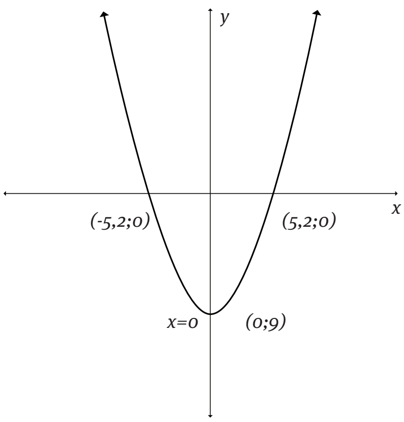
1.2 f: x = 0, g: x = 3, h: x = 2,5, i = x = 4

1.3 g: the axis of symmetry has moved 3 units to the right, p = -3

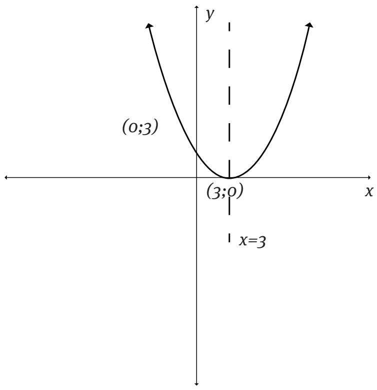
h: the axis of symmetry has moved 2,5 units to the left, p = 2,5

i: the axis of symmetry has moved 4 units to the right, p = -4

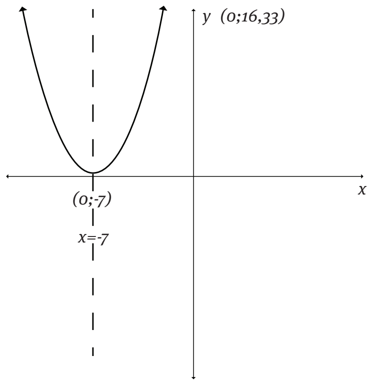
2 2.1



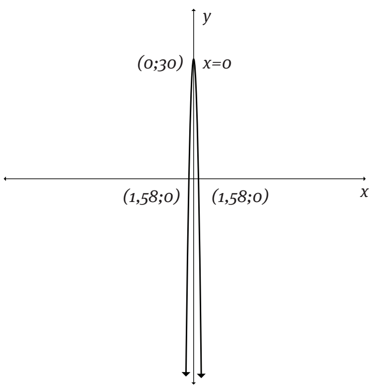
2.2



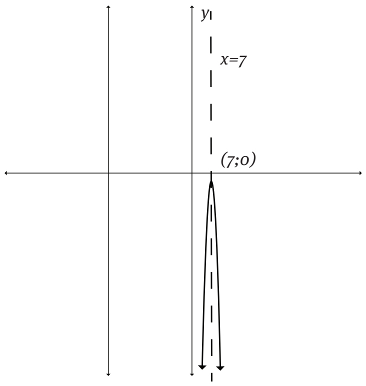
2.3



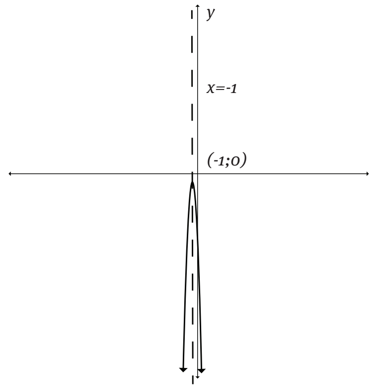
2.4



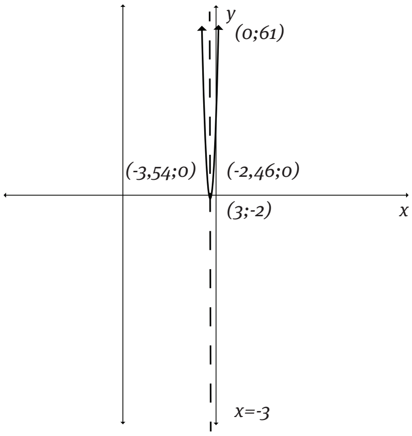
2.5



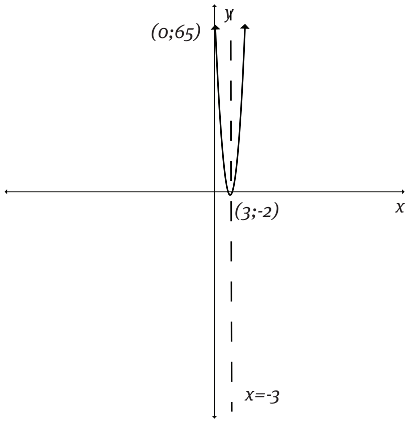
2.6



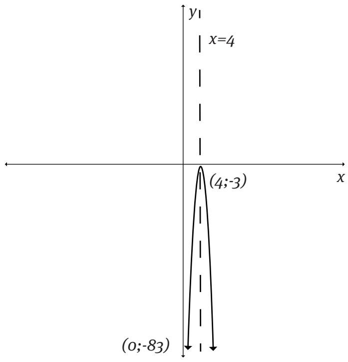
2.7



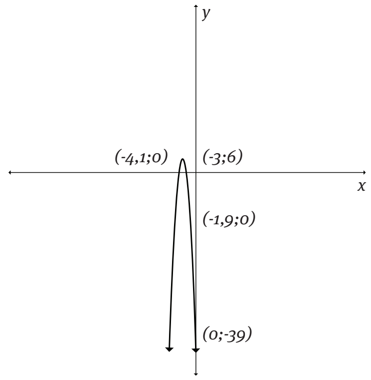
2.8



2.9



2.10



3 3.1 3.1.1i $h(x) = x^2 - 5$

$h(3) = (3)^2 - 5$

$= 4$

3.1.2 $h(x-3) = (x-3)^2 - 5$

$= x^2 - 6x + 9 - 5$

$= x^2 - 6x + 4$

3.1.3 $h(x) - 3 = (x^2 - 5) - 3$

$= x^2 - 8$

3.1.4 $h(x-3) + 7 = (x-3)^2 - 5 + 7$

$= x^2 - 6x + 9 - 5 + 7$

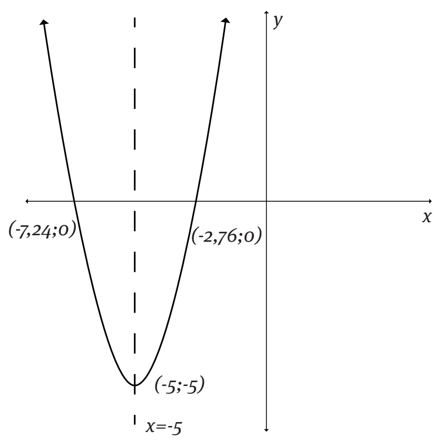
$= x^2 - 6x + 11$

3.2 h(x) is the original function

h(x-3): the graph shifts three units to the right

h(x) -3: the graph shifts three units downwards

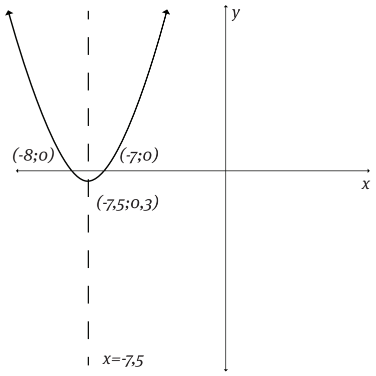
4 4.1



Domain: $x \in R$

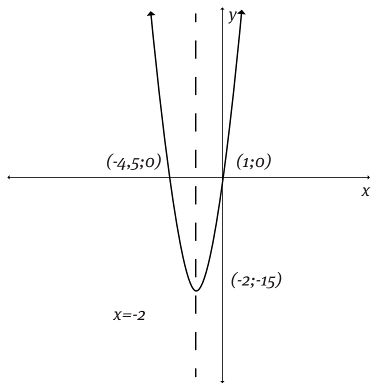
Range: $y \geq -5$

4.2



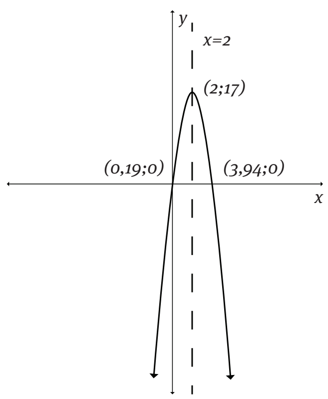
Domain: x ∈ R
Range: y ≥ -0,3

4.3



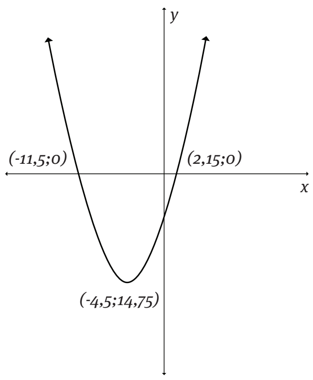
Domain: x ∈ R
Range: y ≥ -15

4.4



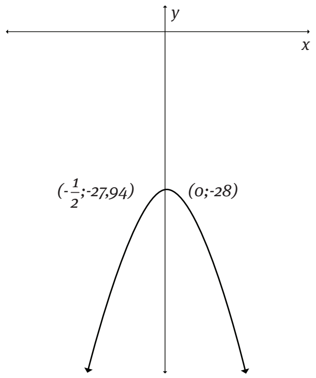
Domain: x ∈ R
Range: y ≤ 17

5 5.1



Domain: x ∈ R
Range: y ≥ -14,75

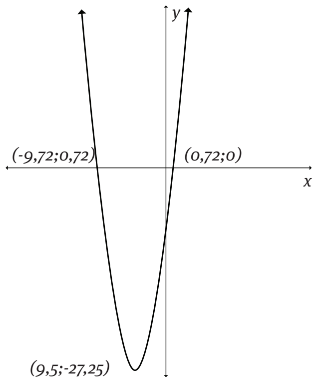
5.2



Domain: $x \in R$

Range: $y \leq -27,94$

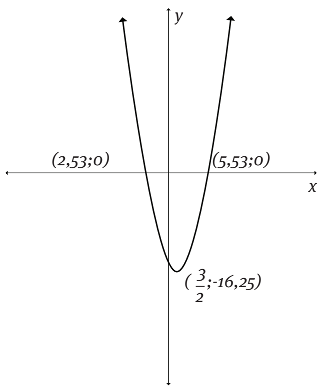
5.3



Domain: $x \in R$

Range: $y \geq -27,25$

5.4



Domain: $x \in R$

Range: $y \geq -16,25$

6 6.1 $y= x^2 -7 +3$

$y= x^2 - 4$

6.2 $y= x^2 -7 -9$

$y= x^2 -16$

6.3 $y= (x+4)^2 -7$

6.4 $y= (x-7)^2 -7$

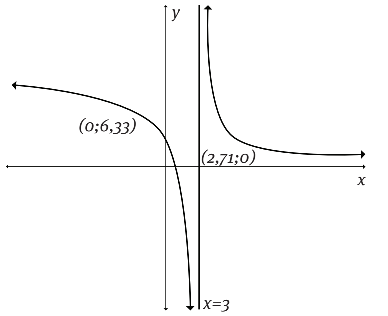
6.5 $y= (x+3/4)^2 -7+ 1/3$

$y= (x+3/4)^2 - 20/3$

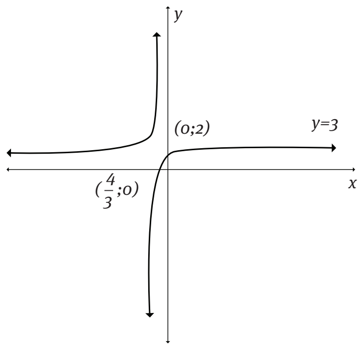
6.6 $y=(x-3,5)^2 -7 -5$

$y= (x-3,5)^2 -12$

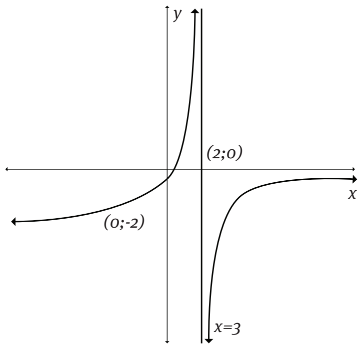
7 7.1



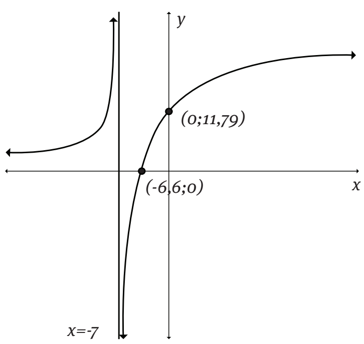
7.2



7.3



7.4



8 f(x) is c, g(x) is a and h(x) is b.

9 9.1 Shifts 4 units upwards
Shifts three units to the left
Shifts 7 units to the right
9.2 Shifts 3 units downwards
Shifts 2 units to the left
Shifts 3 units to the right
9.3 Shifts 5 units upwards
Shifts 1 unit to the right
Shifts 1 unit to the right

10 From the equation $f(x) = (x + p)^2 + q$ it is clear that $a = 1$.
Axis of symmetry is at $x = 2$.
$y = (x + p)^2 + q$
$0 = \left(-\frac{3}{2} - 2\right)^2 + q$
$q = -12{,}25$
$\therefore y = (x - 2)^2 - 12{,}25$

11 $y = \frac{3}{x-6} + 15, y = \frac{-2}{x+4} + 7$ and $y = \frac{-2{,}5}{x+3{,}5} - 5{,}5$

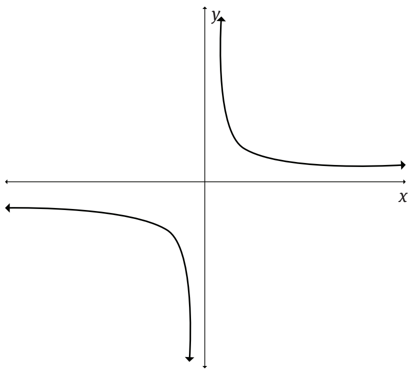
12 $f(x) = \left(-\frac{4}{3}\right)^x, g(x) = 13.5^{-x}$ and $h(x) = -\left(\frac{1}{5}\right)^x + 7$

13 13.1 KL = 2,5 units
13.2 KB = 1 unit
13.3 None
13.4 None

14 14.1 $m_{MN} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-6}{0-0{,}77} = \frac{3}{0{,}77} = 3{,}90$
14.2 $m_{MN} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-2}{-1-6{,}25} = \frac{3}{-7{,}25} = -0{,}41$
14.3 $m_{MN} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-6}{0-0{,}77} = \frac{3}{0{,}77} = 3{,}90$

15 15.1 y= a/x
substitute (1;15)
15=a/1
15= a

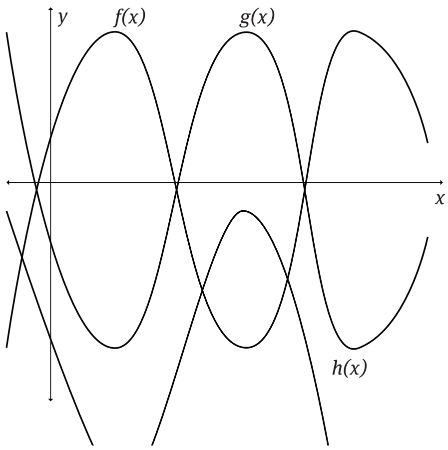
15.2



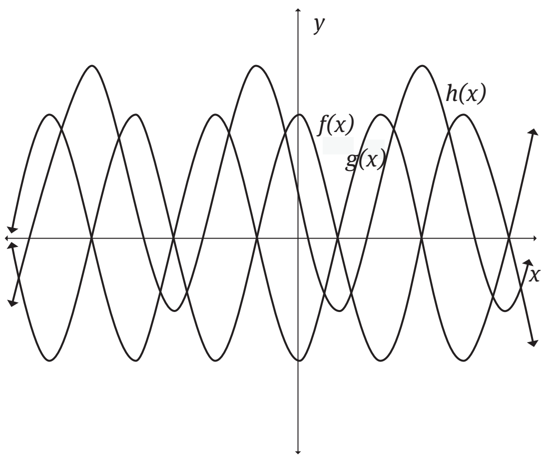
15.3 The graph will shift 15 units upwards

15.4 The restriction is $x \neq 0$, because it is undefined.

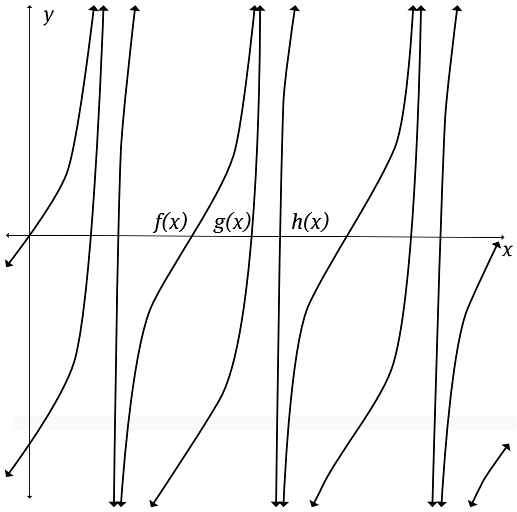
16 16.1



16.2



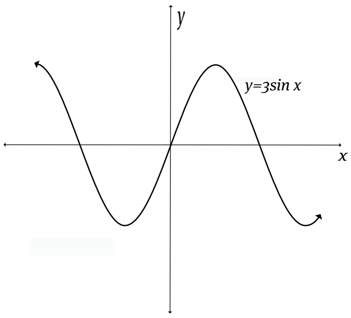
16.3



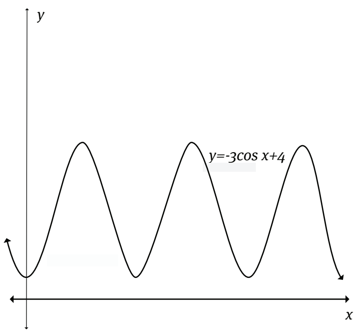
17

	Highest y-value	Lowest y-value	Amplitude	Range	Period
y=3sinx+5	8	2	3	y ε [2;8]	360⁰
y=cosx-5	-4	-6	1	y ε[-6;-4]	360⁰
y=-tanx-3	-2	-4	1	Undefined	180⁰
y=-4cosx+7	11	3	4	y ε[3;11]	360⁰
y=tanx +5	6	4	1	Undefined	180⁰
y= -5sinx+2	7	-3	5	y ε[-3;7]	360⁰

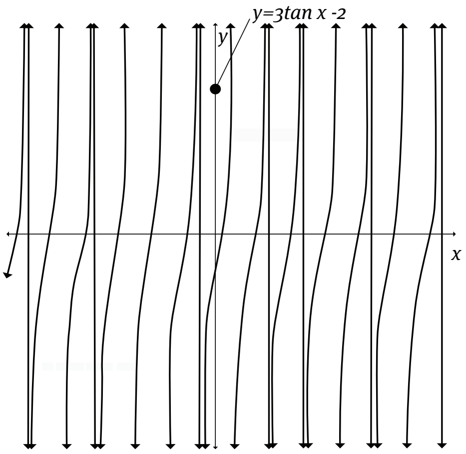
18 18.1



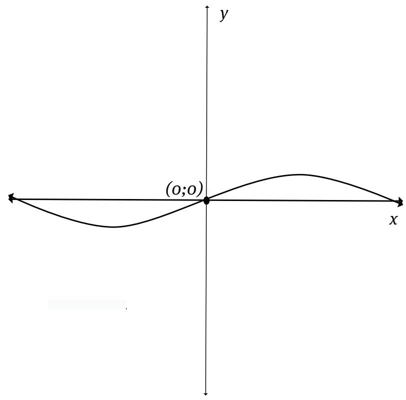
18.2



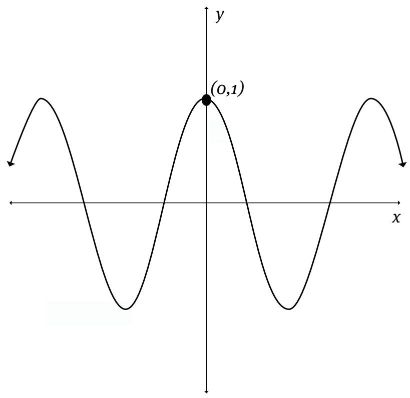
18.3



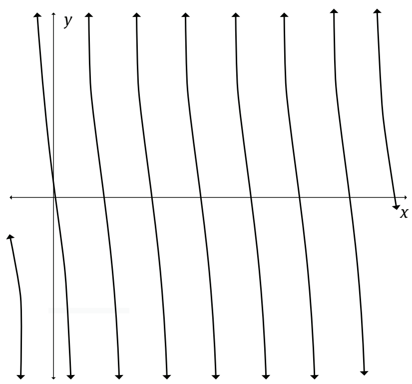
19 19.1 1080^0 , 1, y ε[-1;1]



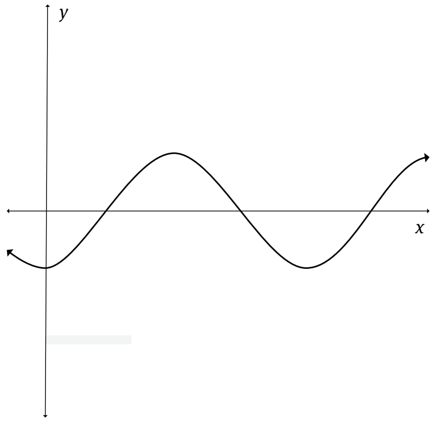
19.2 60^0, 1, y ε[-1;1]



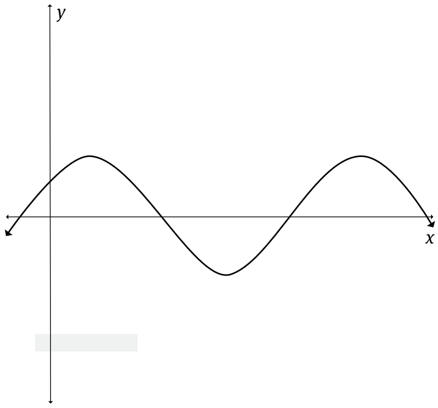
19.3 72^0, 1, undefined



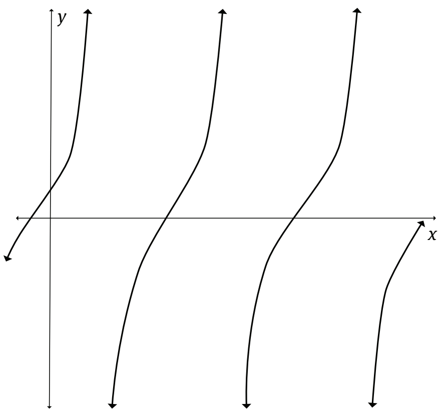
20 20.1



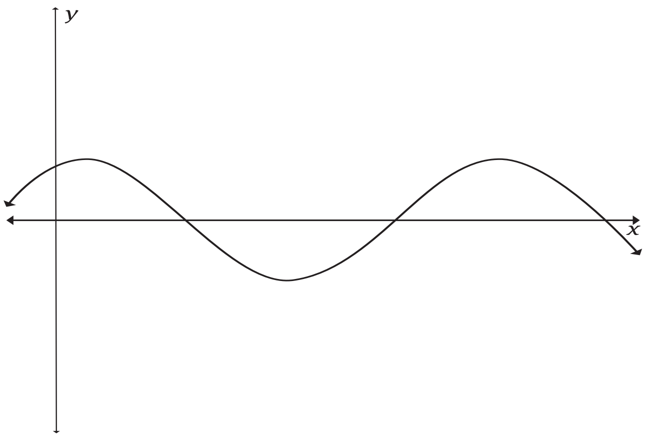
20.2



20.3



20.4



21 A=G

B=E

C=F

22 22.1 y ε [-5;5]

22.2 R(180°; 0,866) and R(270°; 0)

22.3 180^0

22.4 ST = 4,00 units

22.5 h(x) = sin 2(x-75^0) + 2,5

Chapter 6

1 1.1 1.1.1 OA = 13,89

1.1.2 $\sin\alpha = -\frac{12}{13,89}$

1.1.3 $\cos\alpha = \frac{7}{13,89}$

1.1.4 $\frac{\sin\alpha}{\cos\alpha} = \frac{\frac{-12}{13,89}}{\frac{7}{13,89}} = -\frac{12}{7}$

1.1.5 $\tan\alpha = -\frac{12}{7}$

1.2 $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$

1.3 $\sin^2\alpha + \cos^2\alpha = \left(-\frac{12}{13,89}\right)^2 + \left(\frac{7}{13,89}\right)^2 = 1$

2 2.1 √(1- cosx)(1+cosx)

= √(1-cosx)²

=1-cosx

2.2 sinx (sinx + tanx.cosx)

=sinx (sinx + sinx.cosx/cosx)

=sinx (2sinx) =2sin²x

2.3 $(\cos x + \sin x)^2 + (\cos x - \sin x)^2$

$=(\cos^2 x + 2\cos x.\sin x + \sin^2 x) + (\cos^2 x - 2\cos x.\sin x + \sin^2 x)$

$=2\cos^{2x} + 2\sin^2 x$

$=2$

2.4 $\cos^2 y + (\sin^2 y.\ \tan y/\cos y)$

$=\cos^2 y + (\sin^2 y.\ (\sin y/\cos y)/\cos y)$

$= \cos^2 y + (\sin^2 y.\ (\sin y/\cos^2 y)$

$=\cos^2 y + (\sin^3 y/\cos^2 y)$

3 3.1 $x = 90^\circ$ and $x = 270^\circ$

RHS$= 1/\cos t - \sin^2 t/\cos t$

$=(1-\sin^2 t)/ \cos t$

$=\cos^2 t/\cos t$

$=\cos t$

=LHS

3.2 $x = 90^\circ$ and $x = 270^\circ$

RHS$= -\cos^2 b (\tan^2 b +2)$

$= -\cos^2 b (\sin^2 b/\cos^2 b +2)$

$=-\sin^2 b -2\cos^2 b$

$= \sin^2 b - 2(1-\sin^2 b)$

$=-\sin^2 b -2 +2\sin^2 b$

$=\sin^2 b -2$

=LHS

3.3 $x = 90^\circ$ and $x = 270^\circ$

RHS $= \sin^2\beta(1+ \tan^2 \beta)$

$= \sin^2\beta(1+ \sin^2\beta/\cos^2 \beta)$

$= (\sin^2\beta.\cos^2\beta + \sin^4 \beta)/\cos^2 \beta$

$= (\sin^2\beta(\cos^2 \beta +\sin^2 \beta)/\cos^2 \beta$

$= \sin^2\beta/\cos^2 \beta$

$=\tan^2 \beta$

=LHS

3.4 $x = 90^\circ$ and $x = 270^\circ$

LHS$= \tan x - \sin x.\cos x$

$=\sin x/\cos x - \sin x.\cos x$

$=(\sin x-\sin x.\cos^2 x)/\cos x$

$=(\sin x(1-\cos^2 x))/\cos x$

$=\sin x(\sin^2 x)/\cos x$

$=\tan x.\sin^2 x$

=RHS

3.5 $x = 0^\circ$, $x = 180^\circ$ and $x = 360^\circ$

LHS= $\sin x - 1/\sin x$
$=(\sin^2 x-1)/\sin x$
$=-\cos^2 x/\sin x$
=RHS

4 4.1 4.1.1 $\cos 222^0$
$=\cos(180^0+42^0)$
$=-\cos 42^0$
4.1.2 $\sin 48^0$
$=\cos(90^0-42^0)$
$=\cos 42^0$
4.2 4.2.1 $\sin 435^0$
$=\sin(360^0+75^0)$
$=\sin 75^0$
4.2.2 $\cos 285^0$
$=\cos(360^0-75^0)$
$=\cos 75^0$

5 5.1 $\cos 2^0$
$=\cos(90^0-88^0)$
$=\sin 88^0 = t$
5.2 $\tan 178^0$
$=\sin 178^0/\cos 178^0$
$=\sin 2^0/\cos 2^0$
$=\sin(90^0-88^0)/\cos(90^0-88^0)$
$=\cos 88^0/-\sin 88^0$
$=\cos 88^0/-t$
5.3 $\cos 92^0$
$=-\cos(180^0-88^0)$
$=-\sin 88^0$
$=-t$

6 6.1 $\tan(360^0+\theta)$
$=\tan\theta$
6.2 $\cos(90^0-\theta)$
$=\sin\theta$
6.3 $\sin(360^0-\theta)$
$=-\sin\theta$
6.4 $\tan(360^0-\theta)$

$=-\tan \theta$

7 7.1 $\sin 150^0$

$=\sin (180^0-30^0)$

$=\sin 30^0$

$=1/2$

7.2 $\cos 300^0$

$=\cos(360^0-60^0)$

$=\cos 60^0$

$=1/2$

7.3 $\tan 390^0$

$=\tan(360^0+30^0)$

$=\tan 30^0$

$=1/\sqrt{3}$

7.4 $\cos 120^0 + \sin 120^0$

$=\cos(180^0+30^0) + \sin (180^0-60^0)$

$=-\cos 30^0 + \sin 60^0$

$=-\sqrt{3}/2 + \sqrt{3}/2$

$=0$

7.5 $(\sin (90^0-x).\cos(180^0+x).\tan(180^0+x))/\cos(90^0-x).\cos(360^0-x).\tan x$

$=(\cos x.-\cos x.\tan x)/ \sin x.\cos x.\tan x$

$=-\cos x/\sin x$

7.6 $\sin(-\theta)$

$=-\sin\theta$

7.7 $\cos (-360^0-\theta)$

$=\cos\theta$

8 RHS= $\cos (-x) - \sin(x-180^0)$

$=\cos x- (-\sin x(180^0-x)$

$=\cos x-(-\sin x)$

$=\cos x+\sin x$

LHS= $\cos x - \cos x.\tan(-x)$

$=\cos x-\cos x.-\tan x$

$=\cos x+\cos x.\sin x/\cos x$

$=\cos x +\sin x$

LHS=RHS

9 9.1 $\sin x=-3.125$

no solution

9.2 $\cos x-0.986=0$

$\cos x=0.986$
$x=9.6^0 + 360^0n$ or $x= 360^0-9.6^0 + 360^0n$
$x=350.4^0 + 360^0n$

9.3 $3\tan x=2.12$
$\tan x=2.12/3$
$x=35.2^0+180^0n$ or $x=180^0+35.2^0 +180^0n$
$x=215.2^0+360^0n$

9.4 $\tan x=\sin 35^0$
$\tan x=0.57$
$x=29.68^0+180n$ or $x=180^0+29.68^0 +180^0n$
$x=209.68^0 + 180^0n$

9.5 $2\cos x+1=0$
$\cos x=-1/2$
$x=180^0-60^0 +360^0n$ or $x=180^0+60^0+360^0n$
$x=120^0 +360^0n$ $x=240^0+360^0n$

9.6 $\tan 4x=\tan 90^0$
$\tan 90^0$ is undefined

9.8 $\sin(5x+30^0)=\sin x$
$5x+30^0=x+360^0n$
$4x=-30^0+360^0n$ or $4x=360-(-30^0)+360^0n$
$x=-7.5^0+90^0n$ $x=97.5^0 +90^0n$

9.9 $\tan(3x-20^0)= \tan (x+40^0)$
$3x-20^0=x+40^0+180^0n$
$2x=60^0 +180^0n$
$x=30^0 +90^0n$

9.10 $3\sin x=-5\cos x$
$3\tan x=-5$
$\tan x=-5/3$
$x=180^0-59.0^0+ 180^0n$
$x=239.04^0+180^0n$

9.11 $5/9 \cos x=-\sin x$
$9/5=-\tan x$
$-9/5=\tan x$
$x=180^0-60.9^0 +180^0n$
$x=240.9^0 +180^0n$

Chapter 7

1 1.1 $S = 4\prod(12)^2$

$= 156,566$

1.2 $S = 4\prod r^2$

$\therefore 512,157 = 4\prod r^2$

$\therefore r^2 = 40\ 756,15909$

$\therefore r = 201,882$ m

$V = (4/3)\prod r^3$

$= (4/3)\prod(201,882)^3$

$= 845,641\ m^3$

2 2.1 $s = \sqrt{(h^2 + r^2)}$

$= \sqrt{(51,2^2 + 13,5^2)}$

$= \sqrt{2\ 803,69}$

$= 52,950$ cm

2.2 $S = \prod r^2 + \prod rs$

$= \prod(13,5)^2 + \prod(13,5)(52,950)$

$= 2\ 818,244\ cm^2$

2.3 $V = (1/3)\prod r^2h$

$= (1/3)\prod(13,5)^2(51,2)$

$= 9\ 771,610\ cm^3$

3 $s = \sqrt{(h^2 + r^2)}$

$= \sqrt{(23^2 + 1,25^2)}$

$= \sqrt{530,5625}$

$= 23,034$ m

$S = 4lh + (1/2)(4ls)$

$= 4(2,5)(23) + (1/2)(2,5)(23,034)$

$= 345,170\ m^2$

4 $S_{hemisphere} = (1/2)(4\prod r^2)$

$= (1/2)(4\prod(13)^2)$

$= 1\ 061,86\ cm^2$

$S_{right\ cylinder} = \prod r^2 + 2\prod rh$

$= \prod(13)^2 + 2\prod(13)(30)$

$= 2\ 981,37\ cm^2$

∴ Surface area of flowerpot: $1\ 061,86 + 2\ 981,37 = 4\ 043,23\ cm^2$

$V_{hemisphere} = (1/2)(4/3)\prod r^2$

$= (1/2)(4/3)\prod(13)^2$

$= 353,95\ cm^3$

$V_{right\ cylinder} = \prod r^2 h$

$= \prod(13)^2(30)$

$= 15\ 927,87\ cm^3$

∴ Volume of flowerpot: $353,95 + 15\ 927,87 = 16\ 281,82\ cm^3$

Chapter 8

1 1.1 NM = MP (perp from centre to chord)

$\therefore \frac{23,5}{2} = 11,75$ cm

1.2 $OP^2 = OM^2 + MP^2$ (Pyth)

$= 13,5^2 + 11,75^2$

$= 320,3125$

∴ OP = 17,90 cm

1.3 Diameter = 2 × OP = 35,8 cm

2 Let OD = x m

$\therefore$ OK = OM = x + 17 (radii)

and KD = DL = 34 m (perp frm cntr)

$\therefore KO^2 = OD^2 + KD^2$ (Pyth)

$\therefore (x + 17)^2 = x^2 + 34^2$

$\therefore x^2 + 34x + 289 = x^2 + 1\,156$

$\therefore 34x = 867$

$\therefore x = 25{,}5$ m

3 In the first circle:

x = 56° and y = 248°

In the second circle:

x = 268° and y = 92°

In the third circle:

x = 36°, y = 36° and z = 288°

4 4.1 x = 70°

$\hat{O}_2 = 290°$

$y = 1/2\ \hat{O}_2 = 145°$

4.2 PL = PM (radii)

$\therefore \hat{L}_2 = \hat{M} = 32°$

$\therefore x = 116°$

$\therefore \hat{P}_2 = \hat{P}_1 = 116°$

PK = PL

$\therefore y = \hat{L}_1 = 32°$

5 $\hat{B}_1 = \hat{C}_4$ (same arc EG)

$= \hat{C}_1$ (vert opp)

$= \hat{B}_4$ (same arc DF)

6 $x = 236°$ (∠s round a point)

$y = 118°$ (∠ at centre = 2∠ on circ)

$z = 118°$ (opp ∠'s of cyclic quad supl)

7 $x = 75°$ (ext ∠'s of cyclic quad)

$y = 103°$ (opp ∠'s of cyclic quad)

8 $H\hat{F}G = 90°$ (∠ in semi circ)

$\therefore x = 180° - (55° + 90° + 12°) = 23°$ (int ∠'s of triangle supl)

9 9.1 $K\hat{N}M = 90°$ (KN⊥ML)

$L\hat{O}M = 90°$ (LO⊥KM)

$\therefore \hat{O}_1 + \hat{N}_2 = 180°$

∴ OPNM cyclic(opp int ∠'s supl)

9.2 $\hat{O}_2 = 90°$ (LO⊥KM)

$= \hat{N}_1$ (KN⊥ML)

∴ KLNO cyclic quad (KL subt = ∠'s)

$\therefore \hat{K}_1 = \hat{L}_2$

10 10.1 $S\hat{L}R = 155° - (S\hat{L}K + R\hat{L}P)$ (adj ∠'s)

$\therefore x = 155° - (90° + 23°)$ (rad OL ⊥ tang KM)

$= 42°$

$S\hat{R}L = 90°$ (∠ in half circ)

	$\therefore y = 50°$	(Int ∠'s of ΔSRL supl)
	$\therefore z = 130°$	(opp int ∠'s cycl quadsupl)
10.2	$x = \hat{L}_2 = \hat{L}_1 = 40°$	(subt by = chords)
	$y = G\hat{L}K = 80°$	(same chork GK)
	$z = \hat{L}_2 = 40°$	(∠ betw tang and chord GH)
10.3	$x = 60°$	(∠ betw tang and chord BF)
	$\hat{B}_2 = 60°$	(base ∠'s of isos triangle)
	$y = 60°$	(int ∠'s of supl)
	$z = 60°$	(∠ betw tang and chord BG)
10.4	$x = \hat{B}_2 = 30°$	(∠ between tang and chord BS)
	$y + x = 48°$	(ext ∠'s of ΔRST)
	$\therefore y = 18°$	
	$R\hat{S}B = 90°$	(∠ in semi-circ)
	$\therefore x = \hat{S}_2 = 72°$	(∠ betw tang and chord BG)

11	$\hat{Z} = \hat{V}_1$	(∠ betw tang WV and chord YV)
	$= \hat{V}_2$	(vert opp ∠'s =)
	$= \hat{T}$	(∠ betw tang XV and chord UV)
	or	
	$\hat{Y} = \hat{V}_3$	(∠ betw tang VX and chord ZV)
	$= \hat{V}_6$	(vert opp ∠'s =)
	$= \hat{U}$	(∠ betw tang WV and chord TV)
	$\therefore YZ \parallel TU$	(alt ∠'s =)

Chapter 9

1 1.1 Area of $\Delta RPS = \frac{1}{2}(14)(14)\sin 35°$

$= 56{,}21\ cm^2$

$\hat{R} = \frac{180°-35°}{2} = 72{,}5°$

$\therefore P\hat{S}T = 107{,}5°$

Area of $\Delta PST = \frac{1}{2}(10{,}5)(14)\sin 107{,}5°$

$= 63{,}00\ cm^2$

1.2 Area of $\Delta PQR = \frac{1}{2}(17)(11)\sin 123{,}4°$

$= 78{,}06\ cm^2$

2 Area $\Delta BDC \frac{1}{2} CD.BC.\sin C = 109$ square units

$\frac{1}{2}(24)(19) = 109$

$\sin C = \frac{109}{228}$

$C = 28{,}56°$

$\frac{AB}{19} = \tan 28{,}56°$

$\therefore AB = 19\tan 28{,}56° = 10{,}34$ units

3 3.1 $\frac{\sin D}{33} = \frac{\sin 82°}{42}$

$\sin D = \frac{33\sin 82°}{42}$

$\sin D = 0{,}778067768$

$\therefore D = 51{,}08°$

$\therefore F = 46{,}92°$

$\frac{DE}{\sin 46{,}92°} = \frac{42}{\sin 82°}$

$DE = \frac{42\sin 46{,}92°}{\sin 82°}$

$= 30{,}98$

3.2 $\hat{R} = 78°$

$\frac{ST}{\sin 78°} = \frac{28}{\sin 60°}$

$ST = \frac{28 \sin 78°}{\sin 60°}$

$= 31{,}63\ mm$

$\frac{SR}{\sin 42°} = \frac{31{,}63}{\sin 60°}$

$SR = \frac{31{,}63 \sin 42°}{\sin 60°}$

$= 24{,}44\ mm$

4 4.1 $\frac{EF}{\sin 15°} = \frac{21{,}3}{\sin 128°}$

$EF = \frac{21{,}3 \sin 15°}{\sin 128°} = 6{,}99$

4.2 Area ΔHEF $= \frac{1}{2}(21{,}3)(6{,}99) \sin 37°$

$= 44{,}80\ cm^2$

4.3 $\frac{\sin HFG}{13{,}9} = \frac{\sin 15°}{6{,}99}$

$\sin HFG = \frac{13{,}9 \sin 15°}{6{,}99} = 0{,}514675926$

$\therefore HFG = 30{,}98°$

5 5.1 $AC^2 = 49^2 + 32^2 - 2(49)(32) \cos 58{,}4° = 1\,781.780199$

$\therefore AC = 42{,}21\ cm$

$\frac{\sin A}{49} = \frac{\sin 58{,}4°}{42{,}21}$

$\sin A = \frac{49 \sin 58{,}4°}{42{,}21} = 0{,}988737734$

$A = 81{,}39°$

$\therefore C = 40{,}21°$

5.2 $16^2 = 9^2 + 9^2 - 2(9)(9) \cos J$

$\cos J = \frac{9^2+9^2-12^2}{2(9)(9)} = 0{,}111111111$

$\therefore J = 83{,}62°$

$K = M = 48{,}19°$

6 $w^2 = v^2 + x^2 - 2vx \cos W$

$2vx \cos W = v^2 + x^2 - w^2$

$\cos W = \frac{v^2+x^2-w^2}{2vx}$

7 $AC^2 = 12{,}5^2 + 9{,}3^2 - 2(12{,}5)(9{,}3)\cos 124{,}7° = 375{,}0974892$

$\therefore AC = 19{,}37\ cm$

8 8.1 50° [co-interior ∠'s; PQ||SR]

8.2 $\frac{\sin PRS}{18} = \frac{\sin 50°}{23{,}5}$

$\sin PRS = \frac{18 \sin 50°}{23{,}5}$

$\therefore PRS = 35{,}93°$

8.3 $C\hat{A}D = 94{,}07°$

Area ΔPSR $= \frac{1}{2}(18)(23{,}5)\sin 94{,}07° = 210{,}97\ cm^2$

8.4 Area ΔPQR = Area ΔPSR $= 210{,}97\ cm^2$

$\frac{1}{2}PR.QT =$ Area ΔPQR

$\frac{1}{2}(23{,}5)QT = 210{,}97$

$\therefore QT = 17{,}95\ cm$

9 9.1 $RS = \frac{1}{2}.x.AR\ sin\theta$

9.2 $AR = \frac{1}{2}.x.RS\cos\theta$

10 10.1 $DBA = y$ [alt ∠'s]

$\therefore \frac{AB}{DB} = \cos y$

$\therefore AB = DB\cos y$ ……………(1)

$C = 90° - x$

$\therefore \frac{BD}{\sin(90°-x)} = \frac{15}{\sin(x+y)}$

$\frac{BD}{\cos x} = \frac{15}{\sin(x+y)}$

$\therefore BD = \frac{15\cos x}{\sin(x+y)}$………………(2)

Putting (2) in (1) yields:

$$AB = \frac{15\cos x\cos y}{\sin(x+y)}$$

10.2 $$AB = \frac{15\cos 12^\circ \cos 30^\circ}{\sin(12^\circ+30^\circ)}$$

$$= 18{,}99$$

11 K = 90°

In ΔKLM, $LM = \sqrt{x^2 + 124{,}10}$

In ΔKOB, $LO = \sqrt{x^2 + 3{,}71}$

$OM^2 = LO^2 + LM^2 - 2(LO)(LM)\cos y$

$\therefore 55{,}20 =$

$$\left(\sqrt{x^2+3{,}71}\right)^2 + \left(\sqrt{x^2+124{,}10}\right)^2 - 2\left(\sqrt{x^2+3{,}71}\right)\left(\sqrt{x^2+124{,}10}\right)\cos y$$

$\therefore 2\left(\sqrt{x^4 + 128{,}41x^2 + 460{,}41}\right)\cos y = 2x^2 + 72{,}61$

$\therefore \cos y = \frac{x^2+36{,}31}{\sqrt{x^4+128.41x^2+460{,}41}}$

Chapter 10

1 1.1 This investment yields R171 000,27

1.2 This investment yields R162 501,29

∴ Investment A yields the most after 7 years.

2 A = 999 822; n = 10; i = 3,2%

$\therefore 999\ 822 = P(1 + 0{,}032/12)^{120}$

$\therefore 999\ 822 = P(1{,}376541357)$

∴ P = R726 329,07

3 n = 4; A = 4P; m = 4

$A = P(1 + i/4)^{16}$

$\therefore 4P = P(1 + i/4)^{16}$

$\therefore 4 = (1 + i/4)^{16}$

$\therefore \sqrt[16]{4} = 1 + i/4$

$\therefore \sqrt[16]{4} - 1 = i/4$

$\therefore i = (\sqrt[16]{4})/4 = 0{,}0226 = 2{,}26\%$

4 Let's work backwards to get the initial investment amount.

$3\ 111\ 052 = P(1 + 0{,}075/365)^{2\ 555}$

$\therefore 3\ 111\ 052 = 1{,}690367683P$

$\therefore P = R1\ 840\ 458{,}75$

R1 840 458,75 + R19 201,85 = R1 859 660,60

$\therefore 1\ 859\ 660{,}60 = P(1 + 0{,}12/12)^{24}$

$\therefore 1\ 859\ 660{,}60 = P(1{,}269734649)$

$\therefore P = R1\ 464\ 605{,}70$

$\therefore x = R1\ 464\ 605{,}70$

5 We'll have to work our way backwards again.

$5\ 825\ 156 = P(1 + 0{,}124/12)^{156}$

$\therefore 5\ 825\ 156 = P(4{,}971533121)$

$\therefore P = R1\ 171\ 702{,}14$

$1\ 171\ 702{,}14 = P(1 + 0{,}09/1)^{6}$

$\therefore 1\ 171\ 702{,}14 = P(1{,}677100111)$

$\therefore P = R698\ 647{,}70$

∴ Larisse's initial investment was R698 647,70.

6 $47\ 540{,}04 = 39\ 015{,}43(1 + i/4)^{12}$

$\therefore 1{,}218493299 = (1 + i/4)^{12}$

$\therefore \sqrt[12]{1{,}218493299} = 1 + i/4$

$\therefore 1{,}016604268 = 1 + i/4$

$\therefore i = 0{,}0664 = 6{,}64\%$

7 7.1 $A = 10\ 000(1 + 0.0432/4)^{60}$

$= 19\ 050{,}83$

7.2 $i_{eff} = (1 + 0{,}0432/4)^4 - 1$

$0{,}0439 = 4{,}39\%$

7.3 $A = 10\ 000(1 + 0{,}0439/4)^{60}$

$= 19\ 249{,}74$

No, the effective annual rate gives a higher final value.

8 8.1 3% of 1 650 kg = 49,5 kg per day

$A = P(1 - in)$

$\therefore 0 = 1\ 650(1 - 0{,}03n)$

$\therefore 0 = 1\ 650 - 49{,}5n$

$\therefore n = 33{,}33$

They will be able to give seeds to poor communities for 33 days.

8.2 $A = P(1 - in)$

$\therefore 132 = 1\ 650(1 - 0{,}08n)$

$\therefore 132 = 1650 - 132n$

$\therefore 132n = 1518$

$\therefore n = 11{,}50$

∴ They will have 132 kg maize seed after 11,5 days.

8.3 $A = P(1 - in)$

$\therefore 0 = 1\ 650(1 - i.75)$

$\therefore 0 = 1\ 650 - 123\ 750i$

$\therefore 123\ 750i = 1\ 650$

$\therefore i = 0{,}0133 = 1{,}33\%$

$\therefore$ They must give out 1,33% per day in order for the maize seed to last 75 days.

9 9.1 $A = P(1 - i)^n$

$\therefore A = 2\,605\,252(1 - 0,125)^7$

$= 1\,023\,071,79$

9.2 $650\,000 = 2\,605\,252(1 - i)^4$

$\therefore 0,249496018 = (1 - i)^4$

$\therefore \sqrt[4]{0,249496018} = 1 - i$

$\therefore 0,706750142 = 1 - i$

$\therefore i = 0,2932 = 29,32\%$

10 $A = P(1 - i)^n$

$\therefore 1\,260 = P(1 - 0,15)^4 = 0,52200625P$

$\therefore P = 2\,413,76$

11 11.1 11.1.1 $A = P(1 - i)^n$

$\therefore A = 1\,250\,000(1 - 0,17)^5$

$= 492\,380,08$

11.1.2 $A = 1\,250\,000(1 + 0,0602)^5 = 1\,674\,360,66$

11.1.3 R1 674 360,66 – 492 380,08 = R1 181 980,58

11.2 $A = 75\,000(1 + 0,12/12)^{24} = 95\,230,10$

No they won't. They will still need

R1 181 980,58 – 95 230,10 = R1 086 750,48

12 First, let's see what the computers will cost in 2,5 years:

$A = 45\,000(1 + 0,0726)^{2,5} = 53\,617,55$

Now, let's see what the investment will be worth in 2,5 years:

$A = 8\,000(1 + 0{,}15/52)^{130} = 11\,633{,}65$

$\therefore$ No, they won't have enough money to replace the computers. They will still need R53 617,55 – 11 633,65 = R4 198,39

Chapter 11

1 1.1 S = {(1,H); (2,H); (3,H); (4,H); (5,H); (6,H); (1,T); (2,T); (3,T); (4,T); (5,T); (6,T)}

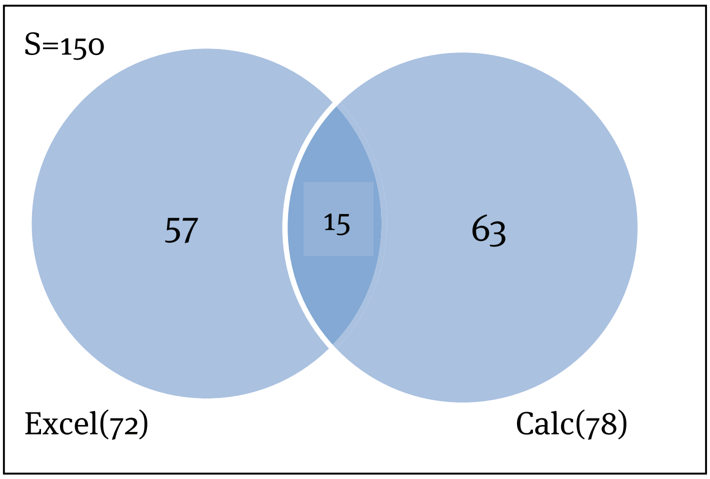
1.2 1.2.1 P(odd and head) = $\frac{3}{12} = \frac{1}{4} = 0{,}25$

1.2.2 P(prime and tail) = $\frac{3}{12} = \frac{1}{4} = 0{,}25$

1.2.3 P(less than 5 and head) = $\frac{4}{12} = \frac{1}{3} = 0{,}333$

1.2.4 P(even and tail) = $\frac{3}{12} = \frac{1}{4} = 0{,}25$

2 2.1



2.2 2.2.1 15 2.2.2 63 2.2.3 57 2.2.4 15

2.3 2.3.1 $\frac{15}{50} = 0{,}1$ 2.3.2 $\frac{63}{150} = 0{,}42$

2.3.3 $\frac{57}{150} = 0{,}38$ 2.3.4 $\frac{15}{50} = 0{,}1$

3 3.1 P(a number less than 4) = $\frac{4}{5} = 0{,}8$

3.2 P(a number greater than 4) = $\frac{1}{5} = 0{,}2$

3.3 P(a number greater than or equal to 4) = $\frac{2}{5} = 0{,}4$

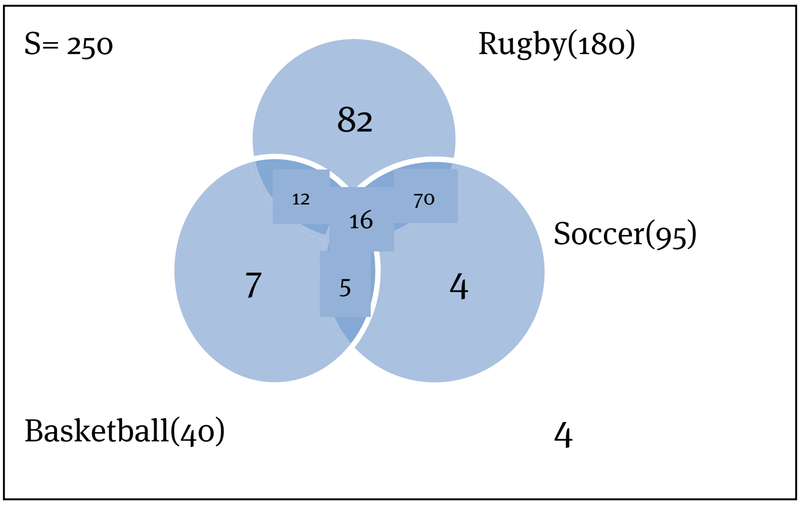
3.4 P(divisible by 3) = $\frac{1}{5} = 0{,}2$

3.5 P(not divisible by 3) = $\frac{4}{5} = 0{,}8$

3.6 3.4 and 3.5 are each other's complements.

3.1 and 3.3 are not each other's complements.

4 4.1



4.2 4 members

4.3 16 members

4.4 41,2%

5 P(A) = $\frac{2}{5}$; P(B) = $\frac{5}{12}$; P(C) = $\frac{1}{3}$

Let's check for mutual exclusivity:

P(A) × P(B) = $\frac{1}{6}$

P(A) × P(C) = $\frac{2}{15}$

P(B) × P(C) = $\frac{5}{36}$

5.1 P(A or C) = P(A) + P(C) – P(A ∩ C)

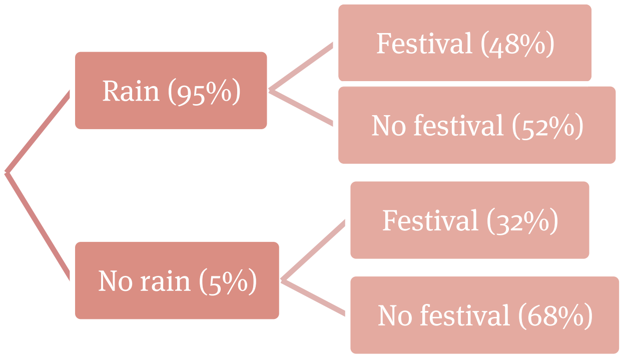
$= \frac{2}{5} + \frac{1}{3} - \frac{2}{15} = \frac{3}{5} = 0{,}6$

5.2 P(A or B) = P(A) + P(B) – P(A ∩ B)

$$\frac{2}{5}+\frac{5}{12}-\frac{1}{6}=\frac{13}{20}=0,65$$

$\therefore 1 - P(A \text{ or } B) = 1 - 0,65 = 0,35$

6 6.1



6.2 6.2.1 P(festival) = $0,95 \times 0,48 + 0,05 \times 0,32 = 0,472 = 47,2\%$

6.2.2 P(no festival in wet weather) = $0,95 \times 0,52 = 0,494 = 49,4\%$

7 7.1

	More than 5 cups	Less than 5 cups	Total
Female	672	1 565	2 237
Male	1 485	1 173	2 658
Total	2 157	2 738	4 895

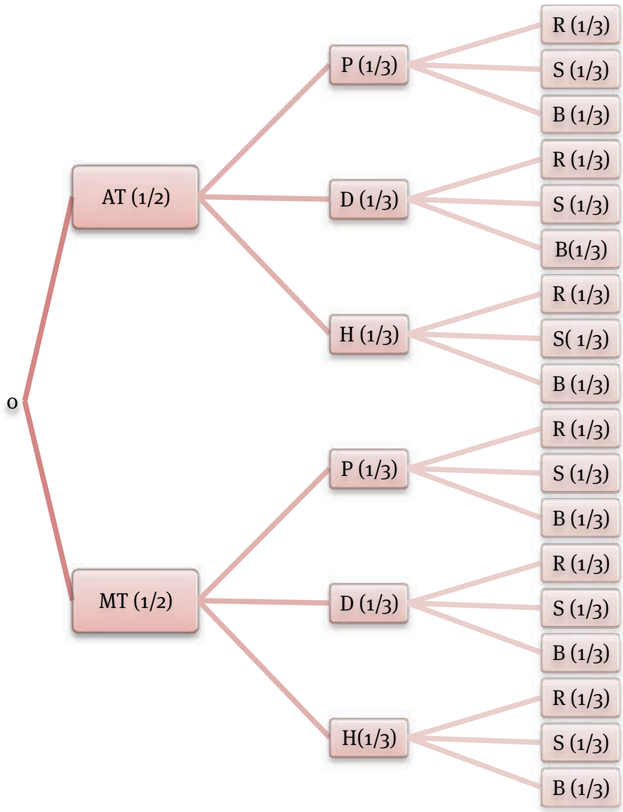
7.2 7.2.1 P(female drinks more than 5 cups a day) = $\frac{672}{4895} = 0,1373$

7.2.2 P(female) = $\frac{2237}{4895} = 0,4570$

7.2.3 P(a person drinks more than 5 cups a day) = $\frac{2157}{4895} = 0,4407$

7.3 own answers.

8 8.1



8.2 8.2.1 P(black) $= \frac{6}{18} = \frac{1}{3}$

8.2.2 P(AT with H) $= \frac{1}{6}$

8.2.3 P(MT with D and R) $= \frac{1}{18}$

8.2.4 P(P with S) $= \frac{1}{3}$

Chapter 12

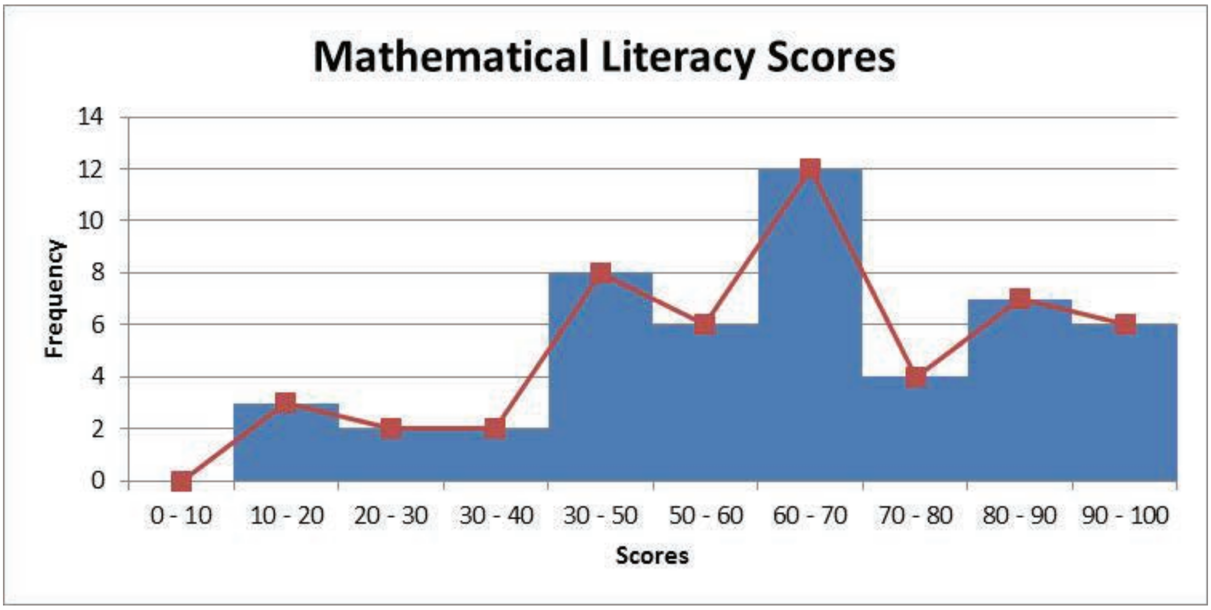
1 1.1

Interval	Frequency
$0 \leq x < 10$	0
$10 \leq x < 20$	3
$20 \leq x < 30$	2
$30 \leq x < 40$	2
$40 \leq x < 50$	8
$50 \leq x < 60$	6
$60 \leq x < 70$	12
$70 \leq x < 80$	4
$80 \leq x < 90$	7
$90 \leq x < 100$	6

1.2 Mode = 82

1.3 The median is in class interval $60 \leq x < 70$.

1.4 and 1.5



2 2.1

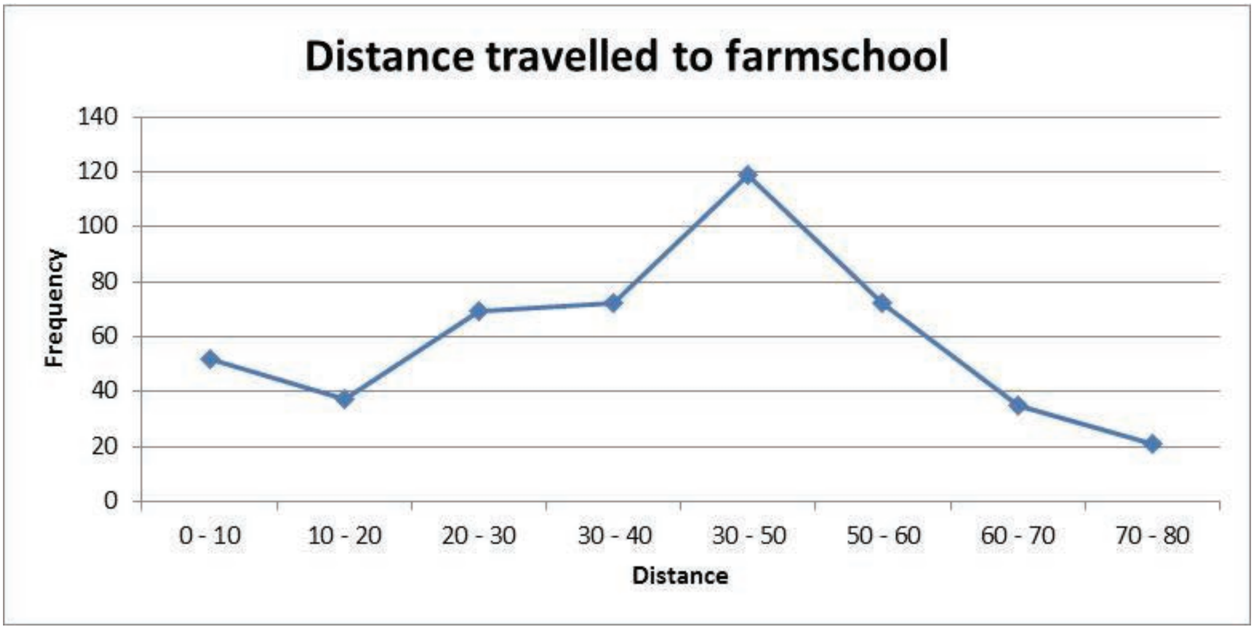
Interval	Frequency
0 ≤ x ‹ 10	52
10 ≤ x ‹ 20	37
20 ≤ x ‹ 30	69
30 ≤ x ‹ 40	72
40 ≤ x ‹ 50	119
50 ≤ x ‹ 60	72
60 ≤ x ‹ 70	35
70 ≤ x ‹ 80	21

Estimated mean:

$$\frac{52\times5+37\times15+69\times25+72\times35+119\times45+72\times55+35\times65+21\times75}{477} = 38{,}21$$

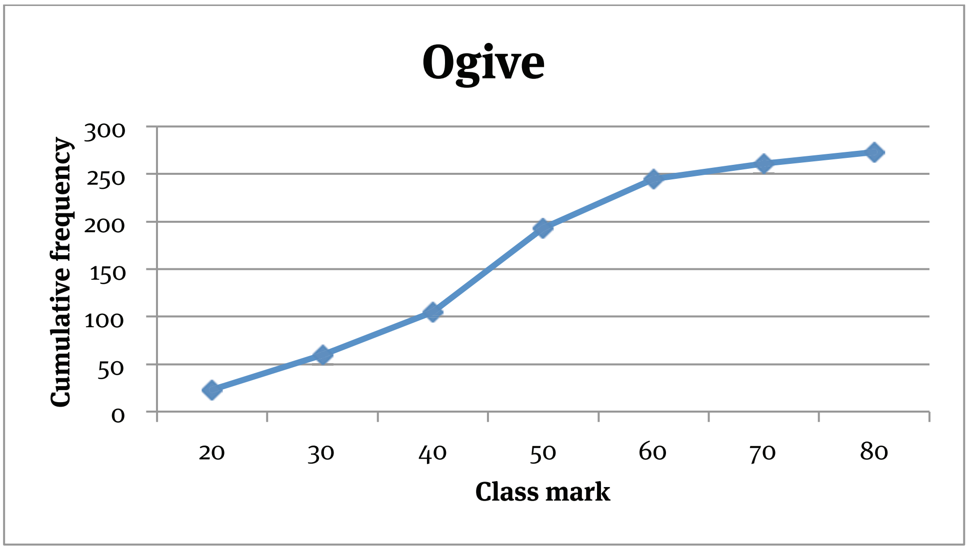
2.2 Modal class = 60 ≤ x < 70

2.3



3

Interval	Frequency	Cumulative frequency	Data point
0 ≤ x ‹ 10	23	23	(20; 23)
10 ≤ x ‹ 20	37	60	(30; 60)
20 ≤ x ‹ 30	45	105	(40; 105)
30 ≤ x ‹ 40	88	193	(50; 193)
40 ≤ x ‹ 50	52	245	(60; 245)
50 ≤ x ‹ 60	16	261	(70; 261)
60 ≤ x ‹ 70	12	273	(80; 273)



4 4.1 124 drivers

4.2 94 drivers

4.3

Interval	Frequency	Cumulative frequency
11 – 15	5	5
15 – 20	25	30
20 – 25	0	30
25 – 30	65	95
30 – 35	21	116
35 - 40	8	124

5 $s^2 = \sum \frac{(x_i - \overline{x})}{n}$

x_i	$x_i - \overline{x}$	$(x_i - \overline{x})^2$
155	0,11	0,0121
142	-12,89	166,1521
169	14,11	199,0921
133	-21,89	479,1721
189	34,11	1 163,4921
128	-26,89	723,0721
175	20,11	404,4121
168	13,11	171,8721

135	-19,89	395,6121

$$\sum x_i = 1\,394$$

$$x_i = 154{,}89$$

$$s^2 = \frac{3702{,}8889}{9} = 411{,}4321$$

$\therefore$ s = 20,2838

6 6.1 6 800

6.2 250

6.3 Less than one, which is not possible, therefore none.

6.4 No.

7 The Mathematics editor is relatively more expensive.

8 8.1 Mean = 130,2 seconds

8.2 $s^2 = \frac{1945{,}6}{10} = 194{,}56$

$\therefore$ s = 13,95

8.3 9,5 ≈ all 10 of the student bookkeepers

9 9.1 Min = 15

Q_1 = 30

Median = 45

Q_3 = 60

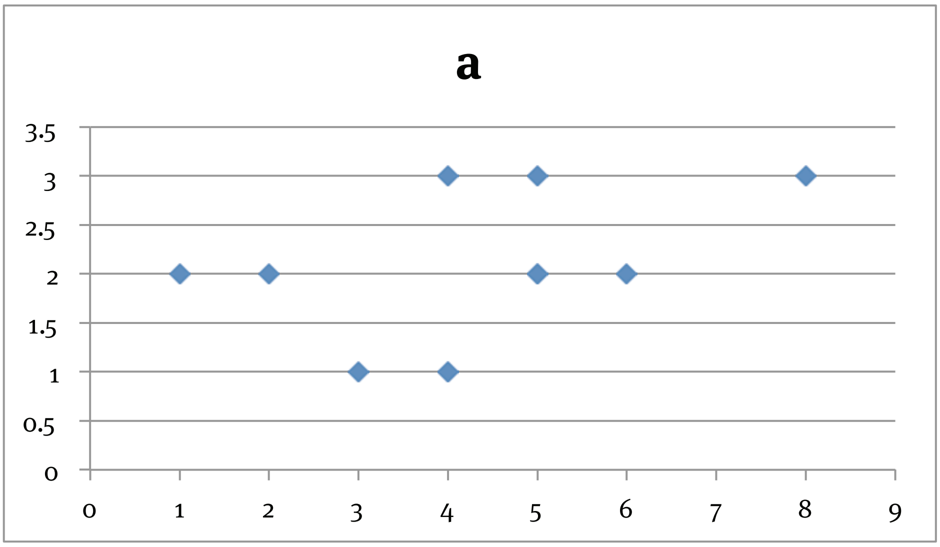
Max = 100

9.2 Semi-quartile range = 156

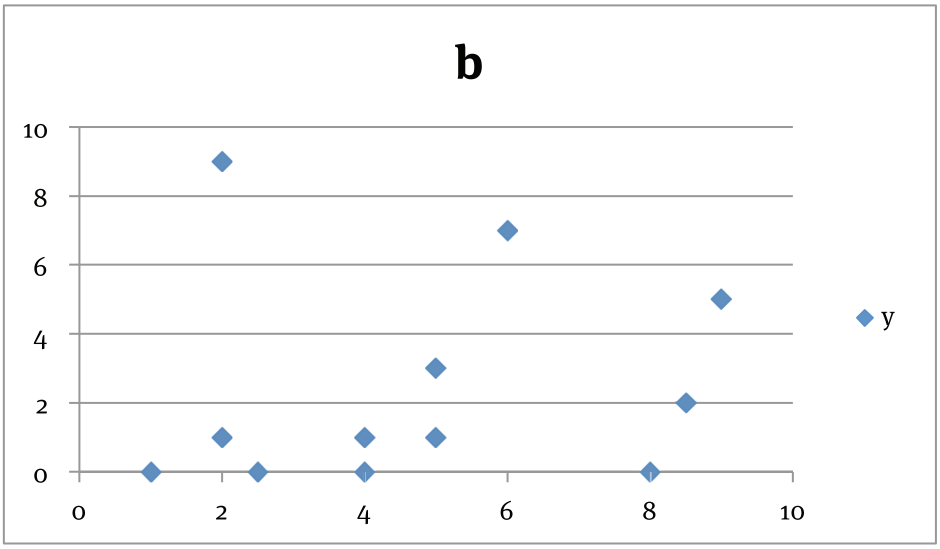
9.3 The data is spread evenly.

9.4 The data is not skewed, it is symmetrical.

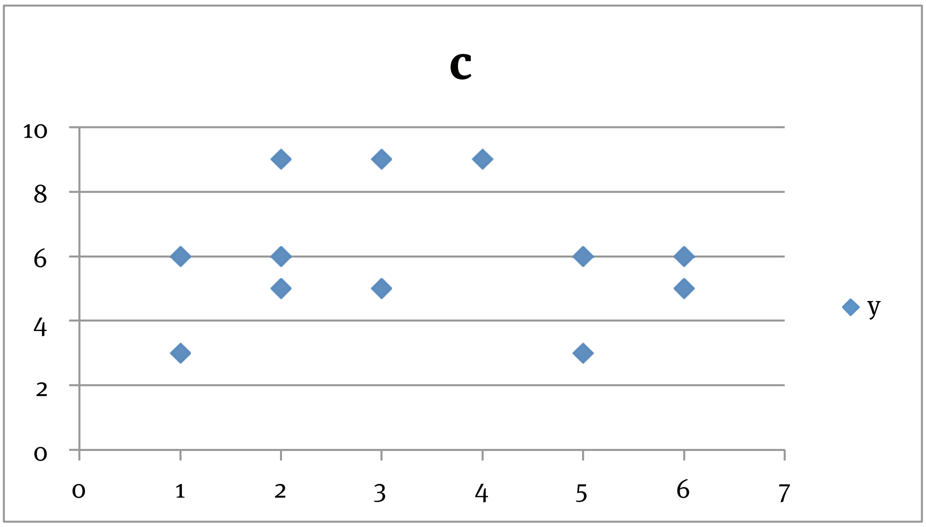
10 10.1



10.2



10.3



Answers to Exemplar Paper 1

QUESTION 1

1.1 1.1.1 $9x(x-6) = 15$

$\therefore 9x^2 - 54x - 15 = 0$

$3x^2 - 18x - 5 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{18 \pm \sqrt{384}}{6}$

$x = 6{,}27$ or $x = -0{,}27$

1.1.2 $x^2 - 4x \geq -4$

$x^2 - 4x + 4 \geq 0$

$(x-2)(x-2) \geq 0$

$x - 2 \geq 0$

$x \geq 2$

1.1.3 $\frac{1}{2}x^2 - 17x = 5$

$\frac{1}{2}x^2 - 17x - 5 = 0$

$x^2 - 34x - 10 = 0$

$x = \frac{34 \pm \sqrt{1196}}{2}$

$x = 34{,}29$ or $x = -0{,}29$

1.2 1.2.1 $x \neq 9$, because that would make the denominator 0, which is undefined.

1.2.2 No. $3^2 = 9$

1.2.3 $\frac{1}{x-9} + 2 = \frac{3x}{9-x}$

$\frac{1}{x-9} + 2 + \frac{3x}{x-9} = 0$

$1 + 2x - 18 + 3x = 0$

$5x = 17$

$x = \frac{17}{5}$

1.3 1.3.1 $2x + 2y = 2 \ldots\ldots\ldots (1)$

$2x^2 + 4y^2 = 4 \ldots\ldots\ldots (2)$

From 1: $2x = 2 - 2y$

$x = 1 - y \ldots\ldots (3)$

(3) in (2): $2(1-y)^2 + 4y^2 = 4$

$2(1 + y^2 - 2y) + 4y^2 = 4$

$2 + 2y^2 - 4y + 4y - 4 = 0$

$2y^2 - 2 = 0$

$y^2 - 1 = 0$

$(y-1)(y+1) = 0$

$y = 1$ or $y = -1$

$y = 1$ in (1): $2x + 2 = 2$ $\qquad$ $y = -1$ in (1): $2x - 2 = 2$

$x = 0$ $\qquad\qquad$ $x = 2$

1.3.2 $4x^2 + 6y - 5 = 0 \ldots\ldots\ldots (1)$
$2x^2 - 7y = 15y \ldots\ldots\ldots (2)$
From (2): $2x^2 = 22y$
$x^2 = 11y \ldots\ldots\ldots (3)$
(3) in (1): $4(11y) + 6y - 5 = 0$
$44y + 6y - 5 = 0$
$50y = 5$
$y = \frac{1}{10} \ldots\ldots\ldots (4)$
(4) in (2): $2x^2 - 7\left(\frac{1}{10}\right) = 15\left(\frac{1}{10}\right)$
$2x^2 = \frac{15}{10} - \frac{7}{10} = \frac{8}{10} = \frac{4}{5}$
$x^2 = \frac{2}{5}$
$x = \frac{2}{5}$ or $x = -\frac{2}{5}$

1.4 The area of the chicken coup increased by a factor of 3,6. The perimeter of the new size chicken coup is $2 \times 1{,}2x + 2 \times 3y = 2{,}4x + 6y$.

QUESTION 2

2.1 2.1.1 $\frac{(4x)^{-3}}{4x^{-2}} = \frac{4x^2}{4x^3} = \frac{1}{4x}$

2.1.2 $\sqrt[3]{512x^{15}} + \sqrt[3]{27x^{15}} = 8x^5 + 3x^5 = 11x^5$

2.1.3 $\sqrt{5}(\sqrt{5} + \sqrt{10}) - \sqrt{4}$
$= 5 + \sqrt{5}\sqrt{10} - 2$
$= 3 + \sqrt{5}\sqrt{10}$

2.2 2.2.1 $4.4^x = 1024$

$4^x = 256 = 4^4$
$x = 4$

2.2.2 $7^x - 7^{x+1} = -42$
$7^x - 7^x.7^1 = -6.7$
$7^x(1 - 7) = -6.7$
$7^x(-6) = -6.7$
$x = 1$

2.2.3 $2.3^{-x} = 54$
$2.3^{-x} = 2.27 = 2.3^3$
$x = -3$

QUESTION 3

3.1 3.1.1 65

3.1.2 $T_n = 2n^2 + 3n$

3.1.3 $T_{16} = 2(16)^2 + 3(16) = 560$

3.1.4 $779 = 2n^2 + 3n$

$2n^2 + 3n - 779 = 0$

$n = \frac{-3 \pm \sqrt{6241}}{4}$

$n = 19$ or $n = -25$ (not possible)

∴The 19th arrangement.

3.2 3.2.1 a = 6; b = 22

3.2.2 $T_1 = 1 = a + b + c$ …………(1)

$T_2 = 6 = 4a + 2b + c$……….(2)

$T_3 = 13 = 9a + 3b + c$ …….(3)

(2) – (1): $5 = 3a + b$

∴ $b = 5-3a$ ………….(4)

(3) – (2): $7 = 5a + b$ ………..(5)

(4) in (5): $7 = 5a + 5 - 3a$

∴ $2 = 2a$

∴ $a = 1$

a in (5): $7 = 5 + b$

a and b in (1): $1 = 1 + 2 + c$

∴ $c = -2$

∴ $T_n = n^2 + 2n - 2$

QUESTION 4

4.1 4.1.1 0 = 350 000(1 – 0,18n)

∴ 0 = 350 000 – 63 000n

∴ 63 000n = 350 000

∴ n = 5,56 ≈ 5 and a half years

4.1.2 $A = 350\ 000(1 - 0{,}25)^4$

= 110 742,19

4.2 Let's assume a loan term of 10 years, and a loan of R10 000.

Loan 1: $A = 10\ 000(1 + 0{,}12/2)^{20} = 32\ 071{,}35$

Loan 2: $A = 10\ 000(1 + 0{,}1/4)^{40} = 26\ 850{,}64$

∴ Loan 2 will cost you less.

4.3 $A = 78\ 000(1 + 0{,}12/12)^{24} = 99\ 039{,}30$

R99 039,30 – 65 000 = R34 039,30

$A = 34\ 039{,}30(1 + 0{,}08/4)^{12} = 43\ 170{,}06$

$A = 43\ 170{,}06(1 + 0{,}19/2)^4 = 62\ 063{,}91$

∴ Danie will have R62 063,91.

QUESTION 5

5.1 (-3; -6)

5.2 $f(x) = -\frac{3}{4}(x+3)^2 - 6 = 0$

$-\frac{3}{4}(x^2 + 6x + 9) - 6 = 0$

$-\frac{3}{4}x^2 - 4{,}5x - 12{,}75 = 0$

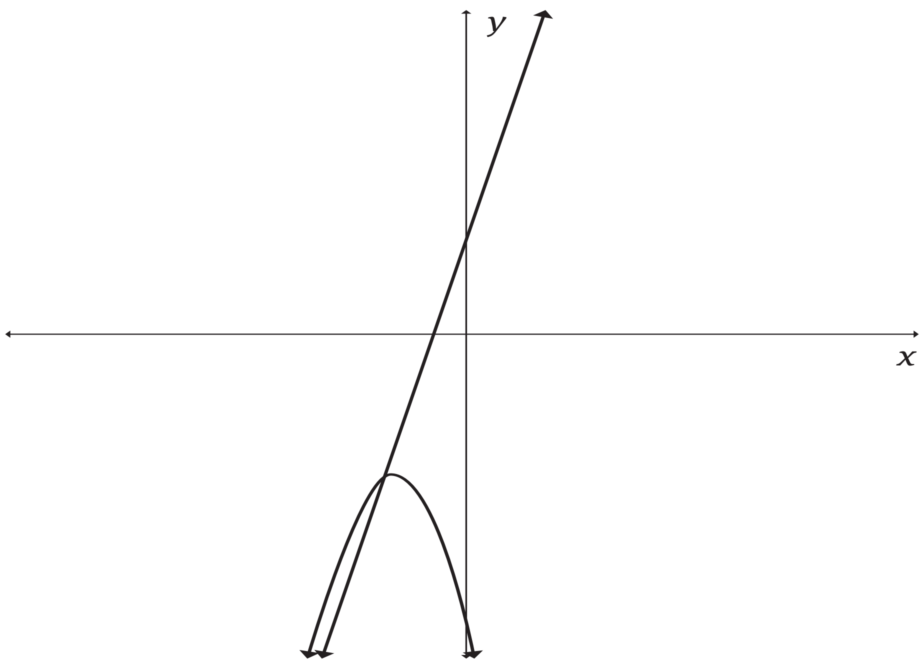
$x^2 + 6x + 17 = 0$

$x = \frac{-6 \pm \sqrt{-32}}{2}$

∴ f(x) has no roots.

5.3 x = -3

5.4



5.5 $f(x+3) = -\frac{3}{4}[(x-3)+3]^2 - 6 = 0$

$= -\frac{3}{4}x^2 - 6 = 0$

∴ $h(x) = -\frac{3}{4}x^2 - 6$

5.6 k(x) = -6x + 5

QUESTION 6

6.1 A(3; -2) and B(1; 5)

6.2 y = -3

6.3 x ∈ R; x ≠ -2

6.4 $r(x) = s(x)$

$\frac{5}{x+2} - 3 = x + 4$

$\frac{5}{x+2} = x + 7$
$5 = (x + 7)(x + 2) = x^2 + 9x + 14$
$x^2 + 9x + 9 = 0$
$x = \frac{-9 \pm \sqrt{45}}{2}$
$\therefore x = -1{,}15$ or $x = -7{,}85$
$r(-1{,}15) = 2{,}88$
$r(-7{,}85) = -3{,}85$

6.5 $-y = x + 4$
$\therefore y = -x - 4$

6.6 $y = \frac{5}{-x+2} + 3$
$= -\frac{5}{x-2} + 3$

6.7 $r(2) = \frac{5}{4} - 3 = -1{,}75$
$\therefore H(2;\ -1{,}75)$

QUESTION 7

7.1 $y = 3^{0-2} = 3^{-2} = \frac{1}{9}$
$\therefore AB = 3 - \frac{1}{9} = 2\frac{8}{9}$

7.2 7.2.1 At point E:
$n(2) = 3^{1-2} = 1/3$
$\therefore$ E(2; 1/3) and A(0; 3)
$\therefore m_{AE} = (3 - 1/3)/(0 - 2) = -1\ 1/3$

7.2.2 At point D:
$m(2) = 3^{2-2} = 1$
$\therefore$ D(2; 1) and B(0; 1/9)
$\therefore m_{BD} = (1/9 - 2)/(0 - 1) = 1\ 8/9$

7.2.3 Both the curves are equally steep.

Answers to Exemplar Paper 2

QUESTION 1

1.1 $m_{RQ} = \frac{-3-3}{-3+6} = -2$

1.2 From point R: 3 = -2(-6) + c
$\therefore$ c = -9
$\therefore$ y = -2x -9

1.3 $m_{SP} = \frac{0-9}{5-3} = -4{,}5$
$\therefore$ They are not parallel, because the gradients are not equal.

1.4 tan θ = 3/8
∴ θ = 20,56°

QUESTION 2

2.1 $BC = \sqrt{(3-3)^2 + (6-1)^2}$
$= \sqrt{25}$
$= 5$

2.2 $m_{AB} = \frac{-2-5}{-2-4} = 1{,}17$
$\therefore\ m_{CD} = -0{,}85$
$\therefore y - 2 = -0{,}85(x-4)$
$\therefore y = -0{,}85x + 5{,}4$

2.3 $AC = \sqrt{(-2-4)^2 + (-2-2)^2}$
$= \sqrt{52}$
$= 7{,}21$
$AC = \sqrt{(-2-4)^2 + (-2-5)^2}$
$= \sqrt{85}$
$= 9{,}22$
This triangle has no special features. It is not an isosceles triangle.

QUESTION 3

3.1 C^BD = 180° - (180° - y + x) = y – x

3.2 $\frac{12}{\sin(y-x)} = \frac{BD}{\sin(90°+x)} = \frac{BD}{\cos x}$
$\therefore BD = \frac{12\cos x}{\sin(y-x)}$

3.3 $\frac{AB}{DB} = \sin y$
$\therefore AB = \frac{12\cos x \sin y}{\sin(y-x)}$

3.4 $AB = \frac{12\cos 43{,}21°.\sin 38°}{\sin(38°-43{,}21°)} = 71{,}30\ m$

QUESTION 4

4.1 Area of pizza $= 6\times(\frac{1}{2}.7.7.\sin 60°) = 127{,}31\ cm^2$

4.2 Side length of 5th pizza: 7 -0,5 – 0,5 – 0,5 – 0,5 = 5 cm
Area $= 6\times(\frac{1}{2}.5.5.\sin 60°) = 64{,}95\ cm^2$

4.3 Volume of hemisphere $= \frac{1}{2}(\frac{4}{3}\pi r^3)$
$= \frac{1}{2}\left(\frac{4}{3}\pi(2)^3\right)$
$= 16{,}76\ cm^3$
Volume of cylinder $= \pi r^2 h$
$= \pi(2)^2.10$
$= 125{,}66\ cm^3$
Total volume = 16,76 + 125,66 = 142,42 cm^3

QUESTION 5

5.1 5.1.1 Lake 1: Min = 12 000 000

Q_1 = 16 000 000 Median = 20 000 000

Q_3 = 22 000 000 Max = 34 000 000

Lake 2: Min = 0

Q_1 = 22 000 000 Median = 28 000 000

Q_3 = 30 000 000 Max = 36 000 000

5.1.2 Lake 2

5.1.3 Lake 1: 6 000 000

Lake 2: 8 000 000

5.1.4 Lake 1 is skewed to the right while lake 2 is skewed to the left.

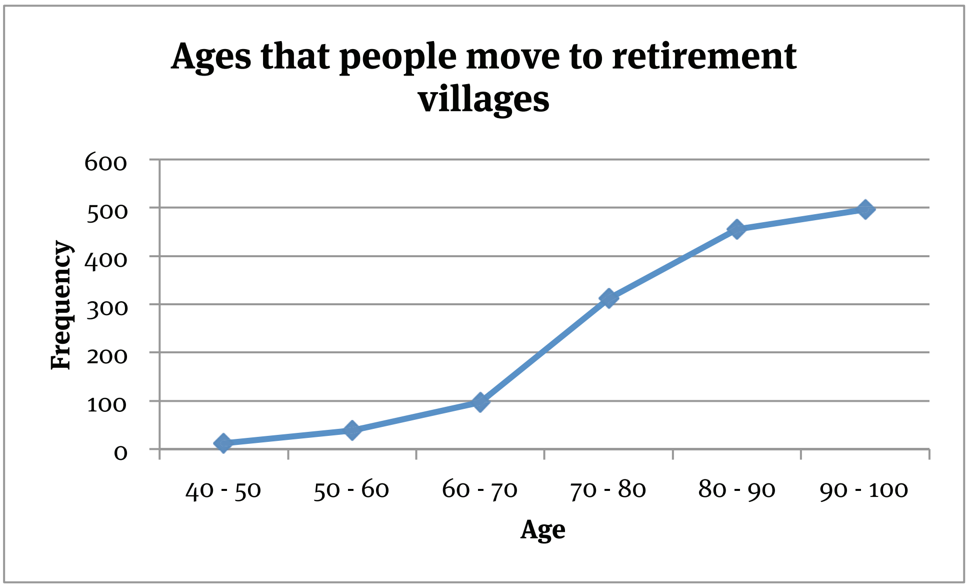
5.1.5 The population in lake 1 is more consistent.

5.2 5.2.1 497 people were questioned

5.2.2

Age	Frequency	Cumulative frequency
$40 \leq x < 50$	12	12
$50 \leq x < 60$	27	39
$60 \leq x < 70$	58	97
$70 \leq x < 80$	215	312
$80 \leq x < 90$	143	455
$90 \leq x < 100$	42	497

5.2.3



5.2.4 The median interval is $70 \leq x < 80$.

QUESTION 6

6.1 6.1.1

Mean $= \frac{2{,}12+2{,}05+2{,}63+1{,}80+1{,}85+2{,}35+2{,}44+2{,}01+3{,}02+1{,}78+1{,}95+1{,}98+2{,}09+2{,}35+1{,}82+2{,}35+1{,}99+1{,}92+2{,}35}{19}$

$= 2{,}15\ m$

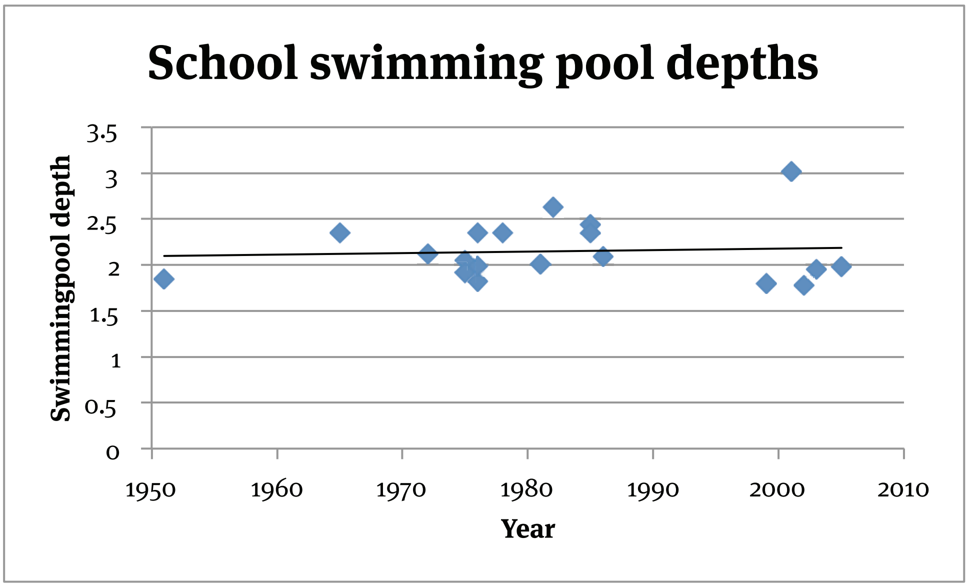
Median = 2,05 m

Mode = 2,35 m

6.1.2 0,31 m

6.1.3 18 swimming pools

6.2 6.2.1



6.2.2 2,1 m

6.2.3 Approximately 1998.

QUESTION 7

7.1 1

7.2 $RHS = \tan x\,(\cos^3 x + \cos x \sin^2 x)$

$= \frac{\sin x}{\cos x}[\cos x(\cos^2 x + \sin^2 x)]$

$= \sin x(1) = \sin x = LHS$

7.3 The solutions will be in the 3rd and 4th quadrants. The reference angle is $\sin^{-1} -0{,}876 = -61{,}16°$. Now let's work out the general solution:

$3x = 180° - 61{,}16° + k.360°$ or $3x = 360° - 61{,}16° + k.360°, k \in R$

$3x = 118{,}83° + k.360°$ $= 298{,}84° + k.360°$

$x = 39{,}61° + k.120°$ $= 99{,}61° + k.120°$

Glossary

radical	The root of a quantity
radicand	The number under the root sign (the radical)
roots	The roots of an equation are the solutions of the equation
zero-product rule	The zero-product principle states "If $a \times b = 0$, then $a = 0$ or $b = 0$"
gradient	A gradient is a slope
inclination	The inclination of a straight line is the angle that the line forms with the positive x-axis
right prism	A right prism is a polyhedron with two identical faces, called the bases
polyhedron	A polyhedron is a geometric solid in three dimensions with flat faces and straight edges
right cylinder	A cylinder with sides perpendicular to the base
axiom	An axiom is a Mathematical statement that is true
theorem	A theorem is a suggestion about something, expressed by formulae, and needs to be proven
corollary	A corollary is a new suggestion that follows from a theorem
converse	The converse is the opposite of something else
circumscribes	A shape that goes around another shape circumscribes the first shape
circumcircle	A circumcircle is a circle that passes through all the vertices of another polygon
circumcentre	If a circle is a circumcircle, then its centre is a circumcentre
cyclic quadrilateral	The vertices of a cyclic quadrilateral all lie on the same circle
constant second difference	When the difference between the first difference is a constant
tangent	A line that touches another curve, but does not cross it
nominal interest	Nominal interest is quoted per year, but compounded more often
quoted interest	Quoted interest is quoted per year, but compounded more often
Venn diagrams	Diagrams that use circles to represent sets and the relationships between them
mutually exclusive	Mutually exclusive events can't be true at the same time
complementary sets	The complement of A consists of all the values not in A. The complement of A is called the complementary set
sample space	The set of all possible outcomes
histogram	A histogram is made up of rectangles with area equal to the frequency of the variable. The width of each rectangle is equal to the class interval
class mark	The midpoint of a class interval
frequency polygon	A line graph used to represent grouped data
ogive	A graph that shows cumulative frequency
cumulative frequency	The cumulative frequency is the sum of all frequencies up to that point
variance	The measure of how far data points are from the mean
standard deviation	The measure of how much the whole set of data points differs from the mean
collinear	If three points are collinear, then they fall on the same straight line

Glossary

angle of elevation	An angle of elevation is the angle between a horizontal line and the line from the observer to some object above the horizontal line
angle of depression	An angle of depression is the angle between a horizontal line and the line from the observer to some object below the horizontal line
true bearing	We measure the true bearing between two points in a clockwise direction starting from north; also referred to as the bearing
conventional bearing	Specifying a bearing using an angle and a compass point as a reference; also called direction
angle	The space between two intersecting lines, usually measured in degrees
degrees	A unit of measurement of angles. There are 360 degrees in a circle
depreciation	The decrease in the value of an asset over time
independent	Does not depend on anything else
dependent	Depends on the value of something else
outliers	An outlier is a value that lies far from the rest of the values of a data set
linear equation	Equation between two variables
standard form	The most common form of an equation or expression